# 164. A Note on the Functional-Representations of Normal Operators in Hilbert Spaces. II

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In this paper we shall discuss the most general type of the functional-representations for normal operators in the abstract Hilbert space $\mathfrak{H}$ which is separable and infinite dimensional.

Lemma A. Let $(\beta_{ij})$ denote any infinite complex matrix

$$\begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \cdot & \cdot & \cdot \\ \beta_{21} & \beta_{22} & \beta_{23} & \cdot & \cdot & \cdot \\ \beta_{31} & \beta_{32} & \beta_{33} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

where $\sum_{i,j=1}^{\infty}|\beta_{ij}|^2<\infty$; and let $B$ denote the operator associated with $(\beta_{ij})$ in Hilbert coordinate space $l_2$. Then, in order that the bounded operator $B$ be normal in $l_2$, it is necessary and sufficient that $\sum_{\nu=1}^{\infty}\beta_{i\nu}\bar{\beta}_{j\nu}=\sum_{\nu=1}^{\infty}\bar{\beta}_{\nu i}\beta_{\nu j}$ for every pair of $i, j=1, 2, 3, \cdots$.

Proof. Since, by hypotheses, $\sum_{i,j=1}^{\infty}|\beta_{ij}|^2<\infty$, it is easily verified with the help of Cauchy's inequality that $||B\tilde{x}||^2\leqq\sum_{i,j=1}^{\infty}|\beta_{ij}|^2\cdot||\tilde{x}||^2$ for every $\tilde{x}\in l_2$. Hence $B$ is a bounded operator in $l_2$. Now we consider the transposed matrix $(\bar{\beta}_{ij})^T$ of $(\bar{\beta}_{ij})$, which is obtained from $(\bar{\beta}_{ij})$ by interchanging rows and columns in $(\bar{\beta}_{ij})$, and denote by $\widetilde{B}$ the operator associated with $(\bar{\beta}_{ij})^T$ in $l_2$. Then, for every pair of elements $\tilde{x}=(x_1, x_2, x_3,\cdots)$ and $\tilde{y}=(y_1, y_2, y_3,\cdots)$ belonging to $l_2$ we have

$$\begin{aligned}(\tilde{x}, \widetilde{B}\tilde{y})&=\sum_{j=1}^{\infty}[\sum_{i=1}^{\infty}\beta_{ij}\bar{y}_i]x_j\\ &=\sum_{i=1}^{\infty}[\sum_{j=1}^{\infty}\beta_{ij}x_j]\bar{y}_i\\ &=(B\tilde{x}, \tilde{y}),\end{aligned}$$

because the absolute convergency of these iterated infinite sums can be verified by virtue of the applications of Cauchy's inequality and the hypothesis $\sum_{i,j=1}^{\infty}|\beta_{ij}|^2<\infty$. Hence $\widetilde{B}$ is the adjoint operator $B^*$ of $B$ in $l_2$. By making use of this result we can readily verify that $BB^*$ is the bounded operator associated with the matrix $(\sum_{\nu=1}^{\infty}\beta_{i\nu}\bar{\beta}_{j\nu})$