106. On the Summability Method (Y)

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§1. When a sequence $\{s_n\}$ is given we consider the transformation

$$(1)\qquad y_n=\frac{1}{2}(s_{n-1}+s_n)\quad(n=0,1,\cdots),$$

where $s_{-1}=0$. If the sequence $\{y_n\}$ tends to a finite limit s, $\{s_n\}$ is said to be summable (Y) to s. This method of summability was studied by O. Szász [4] in detail. G. H. Hardy also remarked this method in his book [1]. As is easily seen, this method is very similar to the ordinary convergence. However, it possesses some interesting properties.

By modifying this method slightly, we obtain the method of summability (Y^*) with the transformation

$$y_n^*=\frac{1}{2}(s_n+s_{n+1})\quad(n=0,1,\cdots).$$

Obviously the methods (Y) and (Y^*) are equivalent. O. Szász [5] proved that the Borel summability (B) does not imply the product summability $(B\cdot Y^*)$.

Recently, W. K. Hayman and A. Wilansky [2] used the method (Y) to construct some counter example. In this note, we shall study these methods furthermore.

§2. We shall prove the following

Theorem 1. *If $\{s_n\}$ is Abel summable (A) to s, then it is also summable $(A\cdot Y)$ to the same sum. Here Y may be replaced by Y^*.*

Proof. The assertion follows from the equality

$$\begin{aligned}(1-x)\sum_{n=0}^{\infty}y_nx^n&=(1-x)\cdot\frac{1}{2}\cdot\sum_{n=0}^{\infty}(s_{n-1}+s_n)x^n\\&=\frac{(1-x)}{2}\left\{\sum_{n=0}^{\infty}s_{n-1}x^n+\sum_{n=0}^{\infty}s_nx^n\right\}\\&=\frac{(1-x)(1+x)}{2}\sum_{n=0}^{\infty}s_nx^n.\end{aligned}$$

In the case of Y^*, the proof is quite similar.

It is interesting to remark that (B) implies[1] $(B\cdot Y)$ but (B) does not imply $(B\cdot Y^*)$ (see O. Szász [5]).

As a converse of the above theorem, we shall prove the following

1) Given two summability methods (P), (Q), we say that (P) *implies* (Q) if any sequence which is summable (P) is summable (Q) to the same sum.