50. On Ascoli-Arzela's Theorem for Metric Space over Topological Semifield

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In this note, we are concerned with the well known theorem of Ascoli and Arzela, and we shall generalize a recent result of K. Vala [4]. We consider a metric space X over a topological semifield K. We denote the metric by d. For the concept of topological semifields, see [1] and [2]. In our discussion, we need the concept of totally boundedness.

Definition 1. A subset A of a metric space X over a topological semifield K is said to be *totally bounded*, if for every neighbourhood U of 0 in K, it is possible to present it as the union of a finite number of sets with diameter less than U, in the other word, given a neighborhood U of 0 in K, there is a finite subset $\{x_k\}$ of X such that, for every $x \in f(A)$, $d(x, x_k) \in U$ for some k.

Let E be an abstract set, then a mapping $f: E \rightarrow X$ is called a *totally bounded mapping*, if $f(E)$ is totally bounded in X. All totally bounded mapping from E to X forms a metric space over K with the metric $\rho(f, g) = \sup_{x \in E} d(f(x), g(x))$. It is evident that each $\rho(f, g)$ is finite, as f, g are totally bounded mapping. This metric space will be denoted by $B_t(E, X)$. We introduce the natural topology in $B_t(E, X)$ by the topological semifield K (see [1] and [2]).

Let H be a subset of the space $B_t(E, X)$. Following K. Vala [4], we call that H has *equal variation* if for any neighborhood U of 0 in K, there is a partition $E_i (i=1, 2, \cdots, n)$ of E such that $x, y \in E_i$ $(i=1, 2, \cdots, n)$ implies $d(f(x), f(y)) \in U$ for every $f \in H$.

Then we have the following result which is a generalization of a theorem by Vala and the idea of its proof is essentially due to K. Vala [4].

Theorem 1. *Let H be a subset of $B_t(E, X)$. H is totally bounded if and only if* 1) *$H(x) = \{f(x) \mid f \in H\}$ for every $x \in E$ is totally bounded and* 2) *H has an equal variation.*

Proof. We shall suppose that H is totally bounded. Then given a neighborhood U of 0 in K, there is a finite subset $f_1, f_2, \cdots, f_n$ of H such that, for each $f \in H$, $\rho(f, f_k) \in U$ for some f_k. For any $x \in E$, we have $d(f(x), f_k(x)) \ll \rho(f, f_k) \in U$, which shows that $H(x)$ is totally bounded. Further, each f_k is totally bounded, so for each k, there is a finite partition $\{E_l^k\} (l=1, 2, \cdots, i_k)$ of E such that $x, y \in E_l^k$