49. On a Criterion of Quasi-boundedness of Positive Harmonic Functions

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1. For a positive[1] harmonic function u on a Riemann surface R, we denote by $\mathfrak{B}u$ the positive harmonic function on R defined by

$$(\mathfrak{B}u)(p)=\sup(v(p);\ u\geqq v,\ v\in HB(R))$$

for p in R. After Parreau we say that u is *quasi-bounded* if $\mathfrak{B}u=u$. In this note we shall give a condition for a positive harmonic function to be quasi-bounded by using the rate of diminishing of harmonic measures of level curves of the harmonic function. For the aim, we set

$$\mathfrak{L}(u;a)=(p\in R;\ u(p)=a)$$

for any positive number a. This is the *a-level curve* of u. For any closed subset F of R, we denote

$$\omega(F;p)=\inf s(p),$$

where s runs over all positive superharmonic functions on R such that $s\geqq 1$ on F. This is the *harmonic measure* of F relative to R calculated at p. Now fix a point p in R. It is clear that $\omega(\mathfrak{L}(u;a);p)=O(1/a)$ for $a\to\infty$. If u is bounded, then $\omega(\mathfrak{L}(u;a);p)=0$ for $a>\sup u$. This suggests us that $\omega(\mathfrak{L}(u;a);p)=o(1/a)$ might be a condition for u to be quasi-bounded. This is really the case and we shall prove

Theorem. *For a positive harmonic function u on a Riemann surface R, the following three conditions are mutually equivalent*:

(1) *u is quasi-bounded on R*;

(2) $\lim_{a\to\infty} a\omega(\mathfrak{L}(u;a);p)=0$ *for some (and hence for any) point p in R*;

(3) $\liminf_{a\to\infty} a\omega(\mathfrak{L}(u;a);p)=0$ *for some (and hence for any) point p in R.*

2. It is clear that the condition (2) implies the condition (3). Hence we have only to show the implications (1)→(2) and (3)→(1). In each case, we may assume that u is unbounded on R and $R\notin O_{HP}$.

Proof of the implication (1)→(2). Fix a point p in R and let R_a be the connected component of the open set $(q\in R;\ u(q)<a)(a>u(p))$ containing the point p. Clearly $\bigcup_{a>u(p)} R_a=R$. Let R^* be the Wiener compactification[2] of R, $\Delta=R^*-R$ and μ be the harmonic measure[2] on

1) By positive, we mean non-negative.

2) C. Constantinescu-A. Cornea: Ideale Ränder Riemannscher Flächen. Springer (1963).