99. A Generalized Derivative and Integrals of the Perron Type

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1. Introduction. Many kinds of integration of the Perron type have been given by various authors using various types of generalized derivatives.

The aim of this paper is to introduce axiomatically a generalized derivative which includes ordinary derivative, approximate derivative and Cesàro derivative, and to build an integral of the Perron type including ordinary Perron integral, AP- and CP-integral defined by J. C. Burkill [1], [2] and more generally G. Sunouchi and M. Utagawa's generalized Perron integral [4], [3].

2. A generalized derivative. Definition 2.1. Let M be a linear space of measurable functions defined on closed interval $[a, b]$. If we can assign uniquely the extented real value $\underline{GD}\,f(x)$ to any function $f(x)\in M$ and any point $x\in[a, b]$ such that

(i) $\underline{GD}\,1=0$,

(ii) $\underline{GD}\,[f(x)+g(x)]\geqq\underline{GD}\,f(x)+\underline{GD}\,g(x)$,

(iii) if $f(x)$ is ordinary differentiable at x then
$$\underline{GD}\,[f(x)+g(x)]=Df(x)+\underline{GD}\,g(x),$$

(iv) $\underline{GD}\,f(x)\geqq\underline{D}\,f(x)$

where $\underline{D}\,f(x)$ means ordinary lower derivate of f at x.

(v) $\underline{GD}\,[\alpha f(x)]=\alpha\,\underline{GD}\,f(x)\qquad(\alpha>0)$,

then $\underline{GD}\,f(x)$ is termed generalized lower derivate of $f(x)$ at x.

Throughout this paper we more assume the following property.

(vi) If $f\in M$ and $\underline{GD}\,f(x)\geqq 0$ at each point x of $[a, b]$ then $f(x)$ is non-decreasing.

Definition 2.2. If we define $\overline{GD}\,f(x)$ by $\overline{GD}\,f(x)=-\underline{GD}[-f(x)]$ then $\overline{GD}\,f(x)$ is called generalized upper derivate of $f(x)$ at x. If $\underline{GD}\,f(x)=\overline{GD}\,f(x)$ then we say that $f(x)$ has generalized derivative at x and the common value is written by $GD\,f(x)$.

Ordinary-, approximate- and Cesàro-lower derivate satisfy the conditions (i)-(vi). The proofs of (vi) for approximate- and Cesàro-derivate were given by G. Sunouchi and M. Utagawa [4].

We can easily prove the following properties.

(1) $\overline{GD}\,1=0$.

(2) $\overline{GD}\,[f(x)+g(x)]\leqq\overline{GD}\,f(x)+\overline{GD}\,g(x)$.