123. Non-negative Integer Valued Functions on Commutative Groups. I

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T. Tamura, one of the authors, introduced "an indexed group" which means a commutative group G with a non-negative integer valued function $I(x, y)$ defined on $G\times G$ and satisfying the following conditions:

(A) $I(x, y)=I(y, x)$

(B) $I(x, y)+I(xy, z)=I(x, yz)+I(y, z)$ for any $x, y, z\in G$

(C) For any $x\in G$, there is a positive integer m (depending on x) such that $I(x^m, x)>0$.

(D) $I(e, e)=1$ where e is the identity of G.

It was shown in [1] that $I(e, x)=1$ for all $x\in G$ for every indexed group G. Consequently if G is periodic, condition (C) is satisfied whenever conditions (A), (B), and (D) are satisfied.

Given an indexed group G, there is a commutative archimedean cancellative semigroup without idempotent such that the fundamental group of which is isomorphic to the group G (Theorem 4 in [1] or Exercise § 4.3, 8. p. 136 in [2]).

The purpose of this paper, as one of the series, is to show how all I-functions on a finitely generated commutative group G may be obtained.

1. The Case where G is a Finite Cyclic Group. Suppose G is a cyclic group of order n generated by a. Let $E(i, j, k)$ denote the equation obtained by setting x, y, z as a^i, a^j, a^k respectively in (B), and let $E'(i, j, k)$ be the equation obtained by exchanging the two sides of $E(i, j, k)$ with each other.

Lemma 1. *$E(m, p, q)$, $m>0$, p, q integers, is expressed by equations of type $E(1, p, q)$.*

Proof. If $m=1$, it is obvious. Let $m\geqq 2$, then $E(m, p, q)$ is obtained by adding $E(m-1, 1, p)$, $E'(m-1, 1, p+q)$, $E(m-1, p+1, q)$ and $E(1, p, q)$. By induction we get this lemma.

For integers $i(\geqq 0), m, n$ we define

$$[m, n]_i=\sum_{k=0}^{i-1} I(a, a^{m+k})-\sum_{k=0}^{i-1} I(a, a^{n-k}), \qquad \text{if} \quad i>0$$
$$=0 \qquad \text{if} \quad i=0$$

Adding then, $E(1, 1, j), E(1, 2, j), \cdots, E(1, i-1, j)$, we obtain