122. *Certain Embedding Problems of Semigroups. II*

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In this paper the author will discuss the problems 1, 3 presented by T. Tamura and N. Graham [4]. The terminology and the numbers of formulas in the previous paper will be used here without definitions.

$\Lambda[P]$ denote the left [right] translation semigroup of a semigroup S. The necessary and sufficient condition so that S is embeddable in the right-sided way was given by Theorem 3 in [4]. But we can find subsemigroups A of Theorem 3 in several ways. We wish to rewrite Theorem 3.

Let C be the set of all left translations λ of S such that $\lambda\rho=\rho\lambda$ for all right translations ρ of S, and D the set of all left translations of S which has a linked right translation of S. If we set $\bar{A}=C\cap D$, then we can prove easily that $\bar{A}$ becomes a semigroup containing the identical mapping $\underline{1}$ and the inner left translation semigroup Λ_0.

If λ and ρ are linked, we write $\lambda(\mathrm{LK})\rho$. As in [4], we have

Lemma a. If $\tau\in\Lambda\cap P$, then it follows that $\tau(\mathrm{LK})\tau$.

Moreover by Lemma a,

Lemma b. If S is embeddable in the mixed way, then $D=\Lambda$.

Let $P\backslash\bar{A}$ be the set of elements of P which are not in $\bar{A}$.

Theorem 3′. S is embeddable in the right-sided way if and only if there exists a left translation $\bar{\alpha}$ of $\bar{A}$ such that $\bar{\alpha}(\mathrm{LK})\rho$ for all $\rho\in(P\backslash\bar{A})$.

Proof. Let S be an embeddable semigroup in the right-sided way. By Theorem 3, there is a subsemigroup A of Λ such that the conditions (15) and (16) hold. If $\alpha\in A$, then there exists $\rho\in P$ where $\alpha(\mathrm{LK})\rho$, and whence $\alpha\in D$. Also we conclude $\alpha\in C$ from (15). Therefore we see that $A\subseteqq\bar{A}$. For every right translation ρ of S, there exists an element α of $A\subseteqq\bar{A}$ such that $\alpha(\mathrm{LK})\rho$ by using (15).

Conversely, if we take $\bar{A}$ as a subsemigroup A of Theorem 3, then $\bar{A}$ satisfies (15) and (16), since we have $\tau(\mathrm{LK})\tau$ for $\tau\in P\cap\bar{A}$.

Theorem 5. Let S be an embeddable semigroup in the mixed way. Then S is embeddable in the right-sided way if and only if every translation ρ in $P\backslash C$ is linked with some left translation γ in C.

Proof. From Theorem 2, if S is embeddable in the mixed way, then there exist subsemigroups A and B having the properties (11), (12), and (13). Let $\tau\in A\cap B$. Then $\tau\beta=\beta\tau$ for all β in B, and $\alpha\tau=\tau\alpha$ for all $\alpha\in A$, and so τ commute with every left and right