110. On Lacunary Trigonometric Series

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§1. **Introduction.** In [2] R. Salem and A. Zygmund proved the

Theorem. *Let* $S_N(t)=\sum_{k=1}^{N} a_k \cos 2\pi n_k(t+\alpha_k)$ *and* $A_N=(2^{-1}\sum_{k=1}^{N} a_k^2)^{1/2}$, *where* $\{n_k\}$ *is a sequence of positive integers satisfying*

(1.1) $\qquad n_{k+1}>n_k(1+c),$ *for some* $c>0$,

and $\{a_k\}$ *an arbitrary sequence of real numbers for which*

$$A_N \longrightarrow +\infty, \text{ and } |a_N|=o(A_N), \qquad \text{as } N\longrightarrow +\infty.$$

Then we have, for any set $E\subset[0, 1]$ *of positive measure and* x,

(1.2) $\lim_{N\to\infty} |\{t;\ t\in E,\ S_N(t)\le xA_N\}|/|E| =(2\pi)^{-1/2}\int_{-\infty}^{x} \exp(-u^2/2)du.$ *)

Recently, it is proved that the lacunarity condition (1.1) can be relaxed in some cases (c.f. [1] and [4]). But in [1] it is pointed out that to every constant $c>0$, there exists a sequence $\{n_k\}$ for which $n_{k+1}>n_k(1+ck^{-1/2})$ but (1.2) is not true for $a_k=1$ and $E=[0, 1]$.

The purpose of the present note is to prove the following

Theorem. *Let* $S_N(t)=\sum_{k=1}^{N} a_k \cos 2\pi n_k(t+\alpha_k)$ *and* $A_N=(2^{-1}\sum_{k=1}^{N} a_k^2)^{1/2}$, *where* $\{n_k\}$ *is a sequence of positive integers satisfying*

(1.3) $\qquad n_{k+1}>n_k(1+ck^{-\alpha})$, *for some* $c>0$ *and* $0\le\alpha\le 1/2$,

and $\{a_k\}$ *an arbitrary sequence of real numbers for which*

(1.4) $\qquad A_N\longrightarrow +\infty$, *and* $|a_N|=o(A_N N^{-\alpha})$, $\qquad$ *as* $N\longrightarrow +\infty$.

Then (1.2) *holds, for any set* $E\subset[0, 1]$ *of positive measure.*

From the above theorem we can easily obtain the

Corollary. *Under the conditions* (1.3) *and* (1.4), *we have*

(1.5) $\qquad \limsup_{N\to+\infty} |\sum_{k=1}^{N} a_k \cos 2\pi n_k(t+\alpha_k)|=+\infty,$ $\qquad$ a.e. *in* t.

For the proof of our theorem we use the following lemma which is a special case of Theorem 1 in [3].

Lemma 1. *Let* $S_N(t)=\sum_{k=1}^{N} a_k \cos 2\pi k(t+\alpha_k)$ *and* $A_N=(2^{-1}\sum_{k=1}^{N} a_k^2)^{1/2}$, *then we put* $\Delta_k(t)=S_{2^{k+1}}(t)-S_{2^k}(t)$ *and* $B_N=A_{2^{N+1}}$. *Suppose if*

$$B_N\longrightarrow +\infty, \text{ and } \sup_t |\Delta_N(t)|=o(B_N), \qquad \text{as } N\longrightarrow +\infty,$$

and

$$\lim_{N\to\infty}\int_0^1 |B_N^{-2}\sum_{K=1}^{N}\{\Delta_K^2(t)+2\Delta_K(t)\Delta_{K+1}(t)\}-1|\,dt=0,$$

then (1.2) *holds, for any set* $E\subset[0, 1]$ *of positive measure.*

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*) $|E|$ denotes the Lebesgue measure of the measurable set E.