149. On Indefinite (E.R.)-Integrals. I

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(Comm. by Kinjirô KUNUGI, M.J.A., Oct. 12, 1965)

§ 1. I.A. Vinogrdova [1] constructed a function $f(x)$ such that (i) $f(x)$ is $\mathcal{D}$-integrable [2] on $[0, 1]$, (ii) $f(x)$ has a continuous indefinite A-integral, $A(x)=(A)\int_0^x f(t)dt$ [3], (iii) $A(x)\neq(\mathcal{D})\int_0^x f(t)dt$ ($x\in P$, mes $P>0$). On the other hand I. Amemiya and T. Ando [4] proved that A-integral is equivalent to (E.R.) integral for Lebesgue measure [5].

In the paper "On indefinite (E.R.)-integrals. II", we shall show that, for every function $f(x)$ which is $\mathcal{D}$-integrable on $I_0=[a, b]$, there exists a measure ν such that $f(x)$ has a indefinite (E.R. ν)-integral, (E.R. ν)$\int_a^x f(t)dt$ [6], and (E.R. ν)$\int_a^x f(t)dt=(\mathcal{D})\int_a^x f(t)dt$ for all $x\in I_0$.

For this purpose, first we shall generalize (see the Lemma of § 2) the theorem which has been proved by S. Nakanishi (formerly S. Enomoto) [7].

Nakanishi's theorem. *Let $f(x)$ be a function which is $\mathcal{D}^*$-integrable [8] on $I_0=[a, b]$ and let $F(I)=(\mathcal{D}^*)\int_I f(x)dx$. Then, for every sequence $\{\varepsilon_n\}$, $\varepsilon_n\downarrow 0$, there exists a non-decreasing sequence of closed sets such that* (i) $\bigcup_{n=1}^{\infty} F_n=I_0$, (ii) *$f(x)$ is summable on every F_n,* (iii) *the condition, $I_i\cap F_n\neq\phi$ for all i, implies that*

$$\left|\sum_{i=1}^{i_0} F(I_i)-\sum_{i=1}^{i_0}(L)\int_{I_i\cap F_n} f(x)dx\right|<\varepsilon_n$$

for every finite family $\{I_i : i=1\cdots i_0\}$ of non-overlapping intervals contained in I_0.

§ 2. For $\mathcal{D}$-integral, we shall prove the following lemma which may be regarded as a generalization of Nakanishi's theorem.

Lemma. *Let $f(x)$ be a function which is $\mathcal{D}$-integrable on $I_0=[a, b]$ and let $F(I)=(\mathcal{D})\int_a^x f(t)dt$. Then, for every sequence $\{\varepsilon_n\}$, $\varepsilon_n\downarrow 0$, there exists a non-decreasing sequence of closed sets $\{F_n\}$ such that* (i) $\bigcup_{n=1}^{\infty} F_n=I_0$, (ii) *$f(x)$ is summable on every F_n,* (iii) $\left|F(I)-\int_{F_n\cap I} f(x)dx\right|\leq\varepsilon_n$ *for every interval $I\subset I_0$,* (iv) $\sum_{i=1}^{\infty}|F(I_n^i)|\leq\varepsilon_n$ *for the sequence of intervals contiguous to the closed set which consists of all points of F_n and end points of I_0.*

Proof. It is enough to show that every function of $\mathcal{L}_\alpha(I_0)$,