177. Axiom Systems of B-algebra

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In my previous note [1], I gave an algebraic formulations of the classical propositional calculus, and I defined four algebraic systems called B, NB, BN, and NBN-algebras. In this note, we shall give other characterizations of B-algebra.

Let $\langle X, 0, *, \sim\rangle$ be an abstract algebra containing 0 as an element of a set X, where $*$ is a binary operation and $\sim$ is an unary operation on X. If $x*y=0$, then we shall denote it by $x\leqslant y$.

The axiom system of B-algebra is given by

$L\,1$ $x*y\leqslant x$,

$L\,2$ $(x*z)*(y*z)\leqslant(x*y)*z$,

$L\,3$ $x*y\leqslant(\sim y)*(\sim x)$,

$L\,4$ $0\leqslant x$.

If $x\leqslant y$ and $y\leqslant x$, then we define $x=y$. This axiom system is equivalent to axioms 1, 2, and

$3'$ $x=\sim(\sim x)$,

$4'$ $(\sim y)*(\sim x)\leqslant x*y$,

as already shown in [1]. Next we shall consider the following axiom system:

$H\,1$ $x*y\leqslant x$,

$H\,2$ $(x*y)*z\leqslant(x*z)*y$,

$H\,3$ $(x*y)*(x*z)\leqslant z*y$,

$H\,4$ $x*(\sim y)\leqslant y$,

$H\,5$ $x*(x*(\sim y))\leqslant x*y$,

$H\,6$ $0\leqslant x$.

We first prove some lemmas from axioms $H\,1\sim H\,6$.

By $H\,2$ and the definition of equality, we have

(1) $$(x*y)*z=(x*z)*y.$$

In $H\,2$, put $x=y, y=y*x, z=y*(\sim x)$, then by $H\,5$, we have

(2) $$x*(x*y)\leqslant x*(\sim y).$$

In (1), put $y=x$, $z=0$, then by $H\,1$, we have

$$(x*x)*0\leqslant(x*0)*x=0,$$

hence $x*x=0$.

(3) $$x*x=0,\text{ i.e. } x\leqslant x.$$

Let $x*z=z*y=0$, then by $H\,3$, we have $x*y=0$. This shows the following

(4) $$x\leqslant y,\ y\leqslant z \text{ imply } x\leqslant z.$$