190. Axiom Systems of B-Algebra

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In his note [1] Kiyoshi Iséki gave an algebraic formulation of propositional calculi and he defined B-algebra.

Other characterisations of B-algebra are given by K. Iséki, Y. Arai, and K. Tanaka (see [2]-[5]).

Let $\langle X, 0, *, \sim\rangle$ be an algebra where 0 is an element of a set X, $*$ is a binary operation and $\sim$ is an unary operation on X. We write $x\leq y$ for $x*y=0$, and $x=y$ for $x\leq y$ and $y\leq x$.

The axiom system of B-algebra is given by (see [2])

$H1.\quad x*y\leq x,$

$H2.\quad (x*y)*z\leq(x*z)*y,$

$H3.\quad (x*y)*(x*z)\leq z*y,$

$H4.\quad x*(\sim y)\leq y,$

$H5.\quad x*(x*(\sim y))\leq x*y,$

$H6.\quad 0\leq x.$

In this note we shall show that a B-algebra is characterized by the following axiom system.

$B1.\quad (x*y)*z\leq x,$

$B2.\quad x*y\leq \sim y,$

$B3.\quad (x*(y*z))*(x*y)\leq x*(\sim z),$

$B4.\quad 0\leq x.$

Lemma 1. $H\Rightarrow B$.

In $H2$, put $z=\sim y$, then by $H4$, we have

$$x*y\leq \sim y, \tag{1}$$

which is axiom $B2$.

In $H3$, put $x*z=z*y=0$, then

$$x\leq y,\ y\leq z \text{ imply } x\leq z. \tag{2}$$

In $H1$, put $x=x*y$, $y=z$, then by $H1$ we have

$$(x*y)*z\leq x*y. \tag{3}$$

By (3), $H1$ and (2) we have

$$(x*y)*z\leq x \tag{4}$$

which is axiom $B1$.

Put $y=y*z$, $z=x*y$ in $H2$, then, we have

$$(x*(y*z))*(x*y)\leq(x*(x*y))*(y*z). \tag{5}$$

Let us put $x=x*z$, $y=y*z$, $z=x*y$ in $H2$, then

$$((x*z)*(y*z))*(x*y)\leq((x*z)*(x*y))*(y*z).$$

The right side is equal to 0 by $H3$, hence we have

$$(x*z)*(y*z)\leq x*y. \tag{6}$$