8. *Algebraic Formulation of Propositional Calculi with General Detachment Rule*

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R. B. Angell [1] formulated a general detachment rule: α and $\varphi(c\alpha\beta)$ imply $\varphi(\beta)$, and further I. Thomas [7] considered on this general detachment rule.

On the other hand, in my notes ([3], [4]), I gave a fundamental idea of algebraic formulations of propositional calculi. This is as follows: Let $M=\langle X, 0, \{o_\alpha\}\rangle$ be an algebra consisting of a set X containing a zero element 0 and a family of operations $\{o_\alpha\}$ containing a binary operation $*$. On the operation $*$, there are common properties: 1) $x*y=0$ is equivalent to $x\leqslant y$, 2) $x=y$ is defined by $x*y=y*x=0$. This means that if $x\leqslant y, y\leqslant x$, then $x=y$.

As easily seen from [1], [7], the general detachment rule is formulated in the form of '$x*0=x$ for all $x\in X$' in the algebra M. Therefore, if we add this axiom to the algebra M, we obtain an algebraic formulation of propositional calculi with a general detachment rule.

In this Note, we shall consider such algebras M.

In our notes ([2], [5]), if an algebra $M=\langle X, 0, *\rangle$ satisfies

1) $(x*y)*(x*z)\leqslant z*y$,
2) $x*(x*y)\leqslant y$,
3) $x\leqslant x$,
4) $x\leqslant 0$ implies $x=0$,

then M is called a BCI-*algebra*.

In the BCI-algebra, we have $(x*y)*z=(x*z)*y$ (see Theorem 1 in [5]). Hence we have $(x*0)*x=(x*x)*0=0$ by 3), and further $x*(x*0)=0$ by 2). This shows $x*0=x$ for all $x\in X$.

Then we have the following

Theorem 1. *An algebra M is a BCI-algebra if and only if M satisfies*

5) $((x*y)*z)*(u*z)\leqslant(x*u)*y$,
6) $x*0=x$,
7) $x\leqslant 0$ *implies* $x=0$.

Proof. Put $z=0$ in 5), then

8) $(x*y)*u\leqslant(x*u)*y$.

Hence we have $(x*y)*u=(x*u)*y$. Next put $y=0$ in 5), then

9) $(x*z)*(u*z)\leqslant x*u$.