3. A Remark on Components of Ideals in Noncommutative Rings

By Hidetoshi MARUBAYASHI

Department of Mathematics, Yamaguchi University

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Let R be a noncommutative ring, let A be an ideal[1] in R, and let M be a non-empty m-system in the sense of McCoy.[2] The right upper and the right lower isolated M-components of A, in the sense of Murdoch,[3] will be denoted by $U(A, M)$ and $L(A, M)$ respectively. In [3], D. C. Murdoch has obtained the following result:

If the ascending chain condition holds in the residue class ring R/A, then $L^n(A, M)$[4] $= U(A, M)$ for some positive integer n.

The aim of this short note is to prove that $n=1$ under an assumption which is weaker than that of Murdoch.

Theorem. *Let $S[A, M]$ be the set of right ideal quotients AB^{-1},[5] where B runs over all ideals meeting the m-system M. Suppose that R satisfies the ascending chain condition for elements of $S[A, M]$. Then $L(A, M)=U(A, M)$.*

Proof. This result will follow from Theorem 5 of [3], if it can be shown that $L(A, M)=L(L(A, M), M)$. By the assumption, there exists a maximal element A_0 in $S[A, M]$ such that $A_0=AB_0^{-1}$ for an ideal B_0 which meets M.

(i) We shall prove that $A_0=L(A_0, M)$. Let x be any element of $L(A_0, M)$. Then we have $xRm\subseteqq A_0$ for some $m\in M$. Hence $x\in A_0(m)^{-1}=A((m)RB_0)^{-1}$, where (m) is the principal ideal generated by m. Now we shall show that $(m)RB_0$ meets the m-system M. For, if $(m)RB_0$ does not meet M, then there exists a prime ideal P, by Lemma 4 of [2], such that $P\supseteqq(m)RB_0$ and $P\cap M=\phi$. Hence we have $m\in P$ or $B_0\subseteqq P$. This is a contradiction. Therefore the maximal property of A_0 implies that $A_0=A_0(m)^{-1}$. Thus $A_0\supseteqq L(A_0, M)$. The converse inclusion is obvious. Hence we have $A_0=L(A_0, M)$.

(ii) We shall prove that $A_0=L(A, M)$. By the definition, we have $A_0\subseteqq L(A, M)$. Let x be any element in $L(A, M)$. Then we have $xRm\subseteqq A$ for some $m\in M$. Thus $xRm\subseteqq A_0$. Hence we obtain $x\in A_0(m)^{-1}$. By the above discussion, it is clear that $A_0=A_0(m)^{-1}$. We have therefore $A_0=L(A, M)$. This completes the proof.

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1) The term *"ideal"* will mean *"two-sided ideal"*.

2) Cf. [2].

3), 4), 5) Cf. [3].