100. *An Integral of the Denjoy Type. III*

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1. **Introduction.** This paper is concerned with the approximately continuous Denjoy integral (AD-integral) defined by the author [3]. The section 2 is devoted to simplify the theory of the AD-integral. The essential point is to use Romanovski's lemma ([2], p. 543). This idea was introduced by S. Izumi [2] who developed the theory of general Denjoy integral very simply using the lemma. In section 3, it will be proved that the AD-integral includes exactly the general Denjoy integral (D-integral) and the approximately continuous Perron integral (AP-integral) defined by J. C. Burkill [1].

2. **The *AD*-integral.** We begin by defining the notion of (ACG). A real valued function $f(x)$ defined on the closed interval $[a, b]$ is said to be (ACG) on the interval if $[a, b]$ is the sum of a countable number of *closed* sets on each of which $f(x)$ is absolutely continuous. Before introducing the AD-integral we need some preparations.

Lemma 1. *If a non-void closed set E is the sum of a countable number of closed sets E_k, then there exists an interval (l, m) containing points of E and an integer k such that $(l, m)\cdot E\subset E_k$.*

For the proof, see, for example, [5], p. 143.

Lemma 2 (**Romanovski**). *Let $\boldsymbol{F}$ be a system of open intervals in $I_0=(a, b)$ such that*

(i) *if $I_k\in \boldsymbol{F}$ $(k=1, 2, \cdots, n)$ and $\left(\bigcup_{k=1}^{n} \bar{I}_k\right)^{\circ}=I$ is an open interval then $I\in \boldsymbol{F}$.*

(ii) *$I\in \boldsymbol{F}$ and $I'\subset I$ imply $I'\in \boldsymbol{F}$.*

(iii) *if $\bar{I}'\subset I$ implies $I'\in \boldsymbol{F}$, then $I\in \boldsymbol{F}$.*

(iv) *if $\boldsymbol{F}_1$ is a subsystem of $\boldsymbol{F}$ such that $\boldsymbol{F}_1$ does not cover I_0, then there is an $I\in \boldsymbol{F}$ such that $\boldsymbol{F}_1$ does not cover I.*

Then $I_0\in \boldsymbol{F}$.

Lemma 3. *If $f(x)$ is absolutely continuous on $[a, b]$ and if $f'(x)=0$ a.e. then $f(x)$ is constant on $[a, b]$.*

Theorem 1. *If $f(x)$ is approximately continuous, (ACG) on $[a, b]$ and if AD $f(x)=0$ a.e. then $f(x)$ is constant on $[a, b]$.*

Proof. Let $\boldsymbol{F}$ be a system of all open intervals of (a, b) in which f is constant. $\boldsymbol{F}$ satisfies evidently the conditions (i), (ii),