203. Some Remarks on Duality Theorems of Lie Groups

By Mitsuo SUGIURA
Department of Mathematics, College of General Education,
University of Tokyo

(Comm. by Zyoiti SUETUNA, M.J.A., Dec. 12, 1967)

1. Introduction. T. Tannaka [1] proved a duality theorem for compact groups. Afterwards C. Chevalley [2] introduced the representative algebra $R(G)$ and the character set $\mathrm{Hom}\,(R(G), \boldsymbol{C})$ and proved Tannaka's theorem for compact Lie groups anew. This work of Chevalley revealed the relation between the compact Lie groups and algebraic groups. The representative algebra $R(G)$ of a general Lie group (not necessarily compact) was studied by G. Hochschild and G. D. Mostow in [3]. They give several conditions each of which is equivalent to say that $R(G)$ is finitely generated. One of these conditions says that the canonical homomorphism maps the connected component G_1 of G onto the connected component of the real proper automorphism group G^* of $R(G)$. This suggests a kind of duality theorem for G.

In this note we say that the duality theorem holds for a topological group G if the canonical homomorphism $\Psi: g \mapsto R_g$ is an isomorphism of G onto the real proper automorphism group G^* of $R(G)$ (cf. 3 for the definitions of G^* and R_g). In 4, we study the relation between our duality theorem and the Tannaka duality theorem (Theorem 1). In 5 we give a necessary and sufficient condition that a Lie group with a finite number of connected components satisfies the duality theorem (Theorem 2). Theorem 2 gives the intimate relation between the duality theorem and the algebraic group structure.

2. The Tannaka duality theorem. Let G be a topological group. In this note, a representation of G means a continuous homomorphism D of G into $GL(n, \boldsymbol{C})$ for some natural number n which is called the degree of D and denoted by $d(D)$. The set of all representations of G is called the dual object of G and denoted by $\mathfrak{R}$. For elements D_1, D_2, and D in $\mathfrak{R}$, the direct sum $D_1 \oplus D_2$, the tensor product $D_1 \otimes D_2$, the equivalent representation $\gamma D \gamma^{-1}$ ($\gamma \in GL(d(D), \boldsymbol{C})$) and the complex conjugate representation $\bar{D}$ are defined as usual. A complex representation ζ of $\mathfrak{R}$ is, by definition, a mapping from $\mathfrak{R}$ into $\bigcup_n GL(n, \boldsymbol{C})$ which satisfies

0) $\zeta(D) \in GL(d(D), \boldsymbol{C})$, 1) $\zeta(D_1 \oplus D_2) = \zeta(D_1) \oplus \zeta(D_2)$,
2) $\zeta(D_1 \otimes D_2) = \zeta(D_1) \otimes \zeta(D_2)$, 3) $\zeta(\gamma D \gamma^{-1}) = \gamma \zeta(D) \gamma^{-1}$