50. *A Characterization of Haar Subspaces in C[a, b]*[*)]

By Yasuhiko IKEBE
Kyoto Sangyo University

(Comm. by Kinjirô KUNUGI, M. J. A., April 12, 1968)

Let M be an n-dimensional subspace of the space $C[a, b]$ with Tchebycheff norm:

$$||f||=\max\{|f(x)|: a\leq x\leq b\}$$

It is well known that the following conditions are mutually equivalent [1]:

(A) If a pair of functions in M agrees on any set of n distinct points in $[a, b]$ then they agree on the entire interval $[a, b]$;

(B) For any basis $\{g_1, \cdots, g_n\}$ of M and for any set of n distinct points $x_1, \cdots, x_n$ in $[a, b]$, the determinant $\det(g_i(x_j))$ is different from 0;

(C) Each element f in $C[a, b]$ has a unique best approximation in M (with respect to the Tchebycheff norm).

Any n-dimensional subspace M of $C[a, b]$ satisfying one of the above conditions (A)—(C) is known as a *Haar* subspace. The purpose of this paper is to show that each one of the above conditions is further equivalent to the following condition:

(D) For each f in $C[a, b]$ which is not identically zero on $[a, b]$ and for each best approximation p in M to f, the following inequality is valid:

$$||p||<2||f||$$

(C)$\Rightarrow$(D). Suppose that (C) is true and let p be a best approximation in M to a non-zero function f in $C[a,b]$. We may assume that $p\neq 0$. Then, from uniqueness,

$$||p-f||<||0-f||$$

and therefore,

$$||p||\leq||p-f||+||f||<||0-f||+||f||=2||f||$$

(D)$\Rightarrow$(B). Suppose that (B) is false. We must show that there exists a nonzero function f in $C[a, b]$ and a best approximation p in M to f

---

*) This paper is part of the author's dissertation written under the supervision of Prof. E. W. Cheney at the University of Texas. This research was supported in part through the Army Research Office (Durham), Project 3772-M, DA-ARO-D-31-124-G388 and G721, and Grants GP-217 and GP-523, National Science Foundation, awarded to the University of Texas, Austin, Texas. Rearrangement for publication was done at IBM Scientific Center, 6900 Fannin, Houston Texas.