# 69. On the Nörlund Summability of the Conjugate Series of Fourier Series

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§ 1. Let $\{p_n\}$ be a sequence such that $P_n=p_0+p_1+\cdots+p_n\neq 0$ for $n=0,1,2,\cdots$. A series $\sum_{n=0}^{\infty} a_n$ with its partial sum $s_n$ is said to be summable $(N, p_n)$ to sum $s$, if $(p_n s_0+p_{n-1}s_1+\cdots+p_0 s_n)/P_n \to s$ as $n\to\infty$. The choice $p_n=1/(n+1)$ yields the familiar harmonic summability. Let $f(t)$ be a periodic finite-valued function with period $2\pi$ and integrable $(L)$ over $(-\pi, \pi)$. Let its Fourier series be

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)\equiv\sum_{n=0}^{\infty}A_n(t). \tag{1.1}$$

Then the conjugate series of the series (1.1) is

$$\sum_{n=1}^{\infty}(b_n\cos nt-a_n\sin nt)\equiv\sum_{n=1}^{\infty}B_n(t). \tag{1.2}$$

Throughout this paper, we write

$$\varphi(t)\equiv\frac{1}{2}\{f(x+t)+f(x-t)-2f(x)\}, \qquad \Phi(t)\equiv\int_0^t|\varphi(u)|\,du,$$

$$\psi(t)\equiv\frac{1}{2}\{f(x+t)-f(x-t)\}, \qquad \Psi(t)\equiv\int_0^t|\psi(u)|\,du$$

and $\tau=[1/t]$, where $[\lambda]$ is the integral part of $\lambda$.

On the Nörlund summability of Fourier series at a given point $x$, the following results are known. Iyengar [3] proved that if

$$\varphi(t)=o(1/\log t^{-1}) \quad \text{as} \quad t\to+0,$$

then the series (1.1) at $t=x$ is harmonic summable to sum $f(x)$. Later, generalizing this result, Siddiqi [5] proved that if

$$\Phi(t)=o(t/\log t^{-1}) \quad \text{as} \quad t\to+0,$$

then the series (1.1) at $t=x$ is harmonic summable to sum $f(x)$. Further, generalizing this result, Pati [7] proved the following

**Theorem A.** *Let $\{p_n\}$ be a sequence such that*

$$p_n>0,\ p_n\downarrow,\ P_n\to\infty \quad \textit{and} \quad \log n=O(P_n).$$

*If*

$$\Phi(t)=o(t/P_\tau) \quad \textit{as} \quad t\to+0,$$

*then the series* (1.1) *at $t=x$ is summable $(N, p_n)$ to sum $f(x)$.*

Furthermore Rajagopal [8] proved the following

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