

2. On a Problem of Vanishing Algebras

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1. Let G be a locally compact group and let m be a left invariant Haar measure on G . For any measurable subset S of G , define L_S to be the subset of $L^1(G)$ consisting of all functions which vanish locally almost everywhere on the complement of S . When L_S forms a sub-algebra of $L^1(G)$, we call it a vanishing algebra. The notion and study of vanishing algebras were initiated by A. B. Simon ([1]).

We see easily

- 1) If S is a semigroup l.a.e., that is, there exists a semigroup T in G such that $G=T$ locally almost everywhere, then L_S is a vanishing algebra.
- 2) If S is a group l.a.e., then L_S is a selfadjoint vanishing algebra.

Teng-Sun Liu proved in [2] that the converse of 2) is always true and the converse of 1) is true when G is unimodular and S is σ -compact. The problem whether the converse of 1) is always true or not seems to be unknown. In this short note we shall give an affirmative answer to this problem by applying essentially the existence of a translation invariant lifting which is due to A. I. Tulcea and C. I. Tulcea ([3]).

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2. The result of [3] says that for an arbitrary locally compact group G there exists a mapping θ of measurable sets into measurable sets such that

- a) $\theta(A)=A$ l.a.e.;
- b) $A=B$ l.a.e. implies $\theta(A)=\theta(B)$;
- c) $\theta(A \cap B)=\theta(A) \cap \theta(B)$;
- d) $\theta(A)=A$ for any open and closed set A ;
- e) $\theta(xA)=x\theta(A)$ for every $x \in G$.

They call it a *lower density commuting with left translations of G* .

Using the above mapping essentially we can show our following answer.

Theorem. *If L_S is a vanishing algebra, then S is a semigroup locally almost everywhere.*

Proof. Suppose $D(S)$ is the set of all points $x \in G$ such that $m(S \cap V) > 0$ for all neighborhoods V of x and suppose $I(S)$ is the set