37. *On a Riemann Definition of the Stochastic Integral.* II

By Sigeyosi OGAWA

Department of Applied Mathematics and Physics,
Faculty of Engineering, Kyoto University

(Comm. by Kinjirô KUNUGI, M. J. A., Feb. 12, 1970)

§ 3. Riemann definition of the stochastic integral.

Now let us get back to the original problem stated in the first paragraph.

As we shall see soon later, the ratio k of interpolation in the Riemann sum (1) performs an important role in this problem, henceforth we shall rewrite the sum $S_n(f)$ into $I_k^{(n)}(f)$ so as to emphasize this ratio k.

Let $\mathcal{S}$ be the class of functions $f_t(\omega)$ which satisfy the conditions $(s,1)$, $(s,2)$ and the following

$(S,3)$ $f_t(\omega)$ is β^+-differentiable in the L_4-sense on $[0,T]$ and its β^+-derivative is Riemann integrable on $[0,T]$ in the L_2-sense.

Now as for the sequence $\{I_k^{(n)}(f)\}(n=1,2,\cdots)$, next Theorem 5 assures the existence of its limit.

Theorem 5. *Let* $\{\Delta^{(n)}\}$ *be a suquence of canonical partitions on* $[0,T]$. *Then for an arbitrary* $f_t(\omega)$ $(\in \mathcal{S})$ *and an arbitrary number* k $(0\leqslant k\leqslant 1)$, *the limit of the sequence exists and is given by*

$$\text{(10)}\qquad \underset{n\to\infty}{\text{l.i.m}}\, I_k^{(n)}(f)(\omega)\equiv I_k(f)(\omega)=I_0(f)(\omega)+k\cdot\int_0^T \frac{\partial^+}{\partial^+\beta_t}f_t(\omega)dt.$$

Definition. We shall call this limit $I_k(f)(\omega)$ the stochastic integral of $f_t(\omega)$ of index k.

As mentioned in § 1, thus constructed stochastic integral $I_k(f)(\omega)$ $(k\neq 0)$ is a generalization of the stochastic integral introduced by R. L. Stratonovich.

For the class of functions which can be represented in the form $\varphi(t,\xi_t(\omega))$ stated in the Example 3, R. L. Stratonovich has defined his integral as the limit in the mean of the following series (see R. L. Stratonovich (2)).

$$\text{(11)}\quad \int_0^T \varphi(t,\xi_t(\omega))d^*\beta_t(\omega)$$
$$\equiv \underset{n\to\infty}{\text{l.i.m}} \sum_i \varphi(t_i^{(n)}, k\xi_{t_{i+1}^{(n)}}+(1-k)\xi_{t_i^{(n)}})(\beta_{t_{i+1}^{(n)}}(\omega)-\beta_{t_i^{(n)}})(\omega)$$
$$t_i^{(n)}\in\Delta^{(n)}\quad\text{and}\quad 0<k\leqslant 1$$

---

*) Rigorously speaking, Stratonovich has defined his integral $\int_0^T \varphi(t,\xi_t(\omega))d\xi_t(\omega)$ only in case of $k=1/2$, in the above definition (11), but the essence of the way of the definition is not much different.