80. *Notes on Modules.* I

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Sharpening a result of Kertész [3], who showed the semisimplicity (in the sense of Jacobson) of the total endomorphism ring $E(M)$ of a completely reducible module M over an arbitrary associative ring A, we prove in our paper the Neumann regularity of this ring $E(M)$. The result also generalizes a theorem of Johnson-Kiokemeister [2], and is an English version of our earlier result [4], written in Hungarian.

Theorem. *Assume that M is a completely reducible module over an arbitrary ring A, and $E(M)$ is the total ring of endomorphisms of M. Then $E(M)$ is regular in the sense of Neumann.*

Proof. Let M be homogeneous. Supposing that the elements of A are right operators, and the elements of $E(M)$ left operators for the module M, for any fixed element $\gamma \in E(M)$ there exists an A-submodule K of M satisfying:

(1) $$M=\gamma M \oplus K$$

being M completely reducible. Denote L_γ the kernel of the endomorphism γ in M, that is

$$L_\gamma=\{m\,;\, m \in M,\ \gamma m=0\}$$

Then L_γ is an A-submodule of M, and there exists another A-submodule N of M with

(2) $$M=L_\gamma \oplus N$$

Being also N completely reducible, we have

$$N=\sum \oplus \{n_\alpha\} \qquad \{\alpha \in \Gamma)$$

with simple A-modules $\{n_\alpha\}$. By (2) our module can be generated by the set of all elements γn_α $(\alpha \in \Gamma)$.

Assume that we have a linear connection

(3) $$\gamma n_{\alpha_1} a_1 + \cdots + \gamma n_{\alpha_k} a_k = 0 \qquad (a_i \in A)$$

then for the element

$$n^* = n_{\alpha_1} a_1 + \cdots + n_{\alpha_k} a_k$$

obviously $\gamma n^*=0$ and $n^* \in L_\gamma$ holds, which yields by (2) also $n^*=0$. The direct sum $\sum \oplus \{n_\alpha\}$ can be built, therefore $n^*=0$ implies $n_{\alpha_1} a_1 = \cdots = n_{\alpha_k} a_k = 0$ and thus also $\gamma n_{\alpha_1} a_1 = \cdots = \gamma n_{\alpha_k} a_k = 0$. Consequently, the set of all γn_α is a basis of γM. Furthermore, let the set of all $k_\beta (\beta \in \Gamma')$ be a basis for k, then by (1) one has

$$M=\sum_{\alpha \in \Gamma} \oplus \{\gamma n_\alpha\} \oplus \sum_{\beta \in \Gamma'} \oplus \{k_\beta\}$$