149. *A Space of Sequences given by Pairs of Unitary Operators*

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(Comm. by Kinjirô KUNUGI, M. J. A., Sept. 12, 1970)

1. Introduction. In a recent note [5] on affine transformations with dense orbits R. Sato makes the following statement (Lemma 1). *Let H be a complex (separable) Hilbert space, and let A be a bounded operator and U_1 and U_2 unitary operators on H. Given $\xi, \eta \in H$ there is a complex, regular Borel measure μ on the two-dimensional torus T^2 whose Fourier-Stieltjes transform is given by*

(1) $$\hat{\mu}(m,n)=\langle AU_1^m\xi, U_2^n\eta\rangle, \qquad -\infty<m,n<\infty.$$

The purpose of this paper is to point out some counterexamples to this proposition and to examine more carefully the class of sequences of the type appearing on the right-hand side of (1).

We refer the reader to [4] for some background and related results on affine transformations. Here let us just recall that doubly-indexed sequences of the type indicated in (1) arise in the study of affine transformations on locally compact groups as follows. Let G be a locally compact group and $\tau(x)$ a *bi*-continuous, Haar-measure-preserving automorphism of G. Let $a \in G$, and consider the affine transformation $T(x)=a\tau(x), x \in G$. Denote the left regular representation of G on $L^2(G)$ by V, and let U_1 and U_2 be the unitary operators on $L^2(G)$ given by composition with $T(x)$ and $\tau(x)$, respectively.

Lemma [5]. $U_2^{-1}VU_1=V\circ T$. *Thus for $f, g \in L^2(G)$ we have*

$$\langle V\circ T^n(x)f, g\rangle=\langle V(x)U_1^n f, U_2^n g\rangle, \qquad -\infty<n<\infty,\ x \in G.$$

The fact that a measure μ satisfying (1) need not exist for all choices of A, U_1 and U_2 is immediate from the following

Proposition. *Let $(a_n)_{n=-\infty}^{\infty}$ be any bounded sequence of complex numbers. There exist a bounded operator A and a unitary operator U on the Hilbert space H and $\xi, \eta \in H$ such that*

$$a_n=\langle AU^n\xi, U^n\eta\rangle, \qquad -\infty<n<\infty.$$

Proof. Let H denote the bilateral sequence space $l_2(-\infty,\infty)$ with standard basis $\{\cdots, e_{-1}, e_0, e_1, \cdots\}$. Let U be the bilateral shift operator: $Ue_n=e_{n+1}, -\infty<n<\infty$, and let A be the bounded operator on H given by coordinatewise multiplication with the given sequence $(a_n)_{n=-\infty}^{\infty}$. Then

*) Research supported by National Science Foundation Grant No. GP-13741.