143. *Bordism Algebra of Involutions*

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(Comm. by Kenjiro SHODA, M. J. A., Sept. 12, 1970)

1. Introduction. Let $\mathfrak{N}_*$ denote the unoriented Thom bordism ring and let $\mathfrak{N}_*(Z_2)$ denote the unoriented bordism group of fixed point free involutions. Then $\mathfrak{N}_*(Z_2)$ is a free $\mathfrak{N}_*$-module with basis $\{[S^n, a]\}_{n\geq 0}$, where $[S^n, a]$ is the bordism class of the antipodal involution on the n-sphere ([2], Theorem 23.2).

If we regard $\mathfrak{N}_*(Z_2)$ as the bordism group of principal Z_2-bundles over closed manifolds, the tensor product of principal Z_2-bundles induces a multiplication in $\mathfrak{N}_*(Z_2)$, making it an algebra over $\mathfrak{N}_*$. Explicitly, we consider involutions T_1 and T_2 on M_1^m and M_2^n respectively, then both $T_1\times 1$ and $1\times T_2$ induce the same involution T on $M_1^m\times M_2^n/T_1\times T_2$. We have then the multiplication

$$[M_1^m, T_1][M_2^n, T_2]=[M_1^m\times M_2^n/T_1\times T_2, T].$$

J. C. Su [6] stated that $\mathfrak{N}_*(Z_2)$ is an exterior algebra over $\mathfrak{N}_*$ with generators in each dimension 2^n $(n=0,1,2,\cdots)$ and C. S. Hoo [4] showed a multiplicative relation in $\mathfrak{N}_*(Z_2)$ which is equivalent to (2.6) below. In this note, we show the following relation.

Theorem. $[S^{2n+1}, a]=[S^1, a]\cdot\left(\sum_{k=0}^{n}[P^{2k}][S^{2n-2k}, a]\right)$ *for all n.*

As an application we show z_{2k} $(k=1,2,3,\cdots)$ in the following result due to Boardman ([1], Theorem 8.1) is nothing else than $[P^{2k}]=[S^{2k}/a]$:

There exist elements $z_2, z_4, z_5, z_6, z_8, \cdots$ in $\mathfrak{N}_*$, uniquely defined by the condition that

$$P=w_1+z_2w_1^3+z_4w_1^5+z_5w_1^6+z_6w_1^7+z_8w_1^9+\cdots$$

(*omitting terms of the form $z_{k-1}w_1^k$ when k is a power of* 2) *is a primitive element in the Hopf algebra $\mathfrak{N}^*(BO(1))$. Moreover, these elements z_k are a set of polynomial generators for $\mathfrak{N}_*$.*

2. Bordism algebra of involutions. Let us summarize here what is known about $\mathfrak{N}_*$-module $\mathfrak{N}_*(Z_2)$. It has been shown that $\mathfrak{N}_*(Z_2)$ is a free $\mathfrak{N}_*$-module with basis $[S^n, a]$ $(n=0,1,2,\cdots)$, where S^n is an n-sphere and a the antipodal involution on S^n. Let

$$\varDelta : \mathfrak{N}_*(Z_2)\to\mathfrak{N}_*(Z_2)$$

be the Smith homomorphism ([2], Theorem 26.1). This is an $\mathfrak{N}_*$-module homomorphism of degree -1, and it can be described as follows. Suppose (M^n, T) is a differentiable fixed point free involution on a