211. *On Semifield Valued Functions*

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In our Note [2], we generalized the well known Hahn-Banach theorem on the extension of linear functionals to semifield valued functions. In this Note, we concern with semifield valued linear functions, and generalize some results of R. P. Agnew [1]. As well known, the theory of semifields has been developed by M. Ya. Antonovski, B. G. Boltjanski and T. A. Sarymsakov since 1960. We shall use some results of semifields without references.

Let E be a linear space, and let F be a semifield. We consider functions defined on E with range in F. A function $r(x)$ is called r-function on E, if there is a linear function $f(x)$ satisfying $f(x) \ll r(x)$ on E.

Let $r(x)$ be an r-function on E. Then there is a linear function $f(x)$ such that $f(x) \ll r(x)$ for all $x \in E$. For $t>0$, we have

$$f(x)=\frac{f(tx)}{t} \ll \frac{r(tx)}{t}$$

Therefore, if $t_1, t_2, \cdots, t_n > 0$, and $\sum_{i=1}^n x_i = 0$, we have

$$0=f(x)=f\left(\sum_{i=1}^n x_i\right)=\sum_{i=1}^n f(x_i) \ll \sum_{i=1}^n \frac{r(t_i x_i)}{t_i},$$

hence

$$0 \ll \inf_{\substack{t_i>0 \\ \sum_{i=1}^n x_i=0}} \sum_{i=1}^n \frac{r(t_i x_i)}{t_i} \tag{1}$$

Conversely, suppose that (1) holds. We define

$$p(x)=\inf_{\substack{t_i>0 \\ x_i=x}} \sum_{i=1}^n \frac{r(t_i x_i)}{t_i}. \tag{2}$$

Then $p(x) \ll r(x)$. Let $\sum_{i=1}^n x_i = x$, then $\sum_{i=1}^n x_i + (-x) = 0$. By the hypothesis, we have

$$0 \ll \sum_{i=1}^n \frac{r(t_i x_i)}{t_i} + r(-x).$$

Hence $-r(-x) \ll p(x)$. If $x \in E$ and $t>0$, then

$$p(tx)=\inf_{\substack{t_i>0 \\ \sum_{i=1}^n x_i=x}} \sum_{i=1}^n \frac{r(t_i x_i)}{t_i}$$