45. *On the Spectrum of the Laplace-Beltrami Operator on a Non-Compact Surface*

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1. Introduction and preliminaries. In this note we consider the Laplace-Beltrami operator $\varDelta$ on a non-compact surface M in the Euclidean 3-space E^3. Our purpose is to show that, under certain assumptions on M, $\varDelta$ has no eigenvalue as an operator in the Hilbert space $L^2(M)$. Several authors have worked on the eigenvalue problems for $\varDelta$ or the Schrödinger operators $-\varDelta+q$ in E^n or in certain unbounded subdomains of E^n. Our problem differs from theirs in that it cannot necessarily be reduced to the problem in the flat Euclidean space. However, suggestions of our method can be found in their works, especially Rellich [1] (see also Eidus [2] p. 42, Theorem 10).

Let M be a surface of class C^2 in E^3. Let (g_{ij}) be the Riemann metric tensor on M, $(g^{ij})=(g_{ij})^{-1}$ and $G=det(g_{ij})$. The Laplace-Beltrami operator $\varDelta$ on M is given by

$$\varDelta u=\sum_{i,j=1}^{2}\frac{1}{\sqrt{G}}\frac{\partial}{\partial x^i}\left(\sqrt{G}\,g^{ij}\frac{\partial u}{\partial x^j}\right)\qquad(\text{for } u\in C^2(M))\tag{1}$$

in any local coordinate system (x^1, x^2). $L^2(M)$ is the totality of measurable functions which are square integrable on M, that is, $L^2(M)$ is a Hilbert space with the scalar product

$$(\phi, \psi)=\int\phi\bar{\psi}dM,$$

where dM is the surface element of M: $dM=\sqrt{G}dx^1dx^2$. D_1 is the completion of $C_0^2(M)$ (C^2 functions on M with compact supports) with regard to the norm

$$\|f\|_1^2=\|f\|^2+\int|\text{grad}\, f|^2\, dM,$$

where $\|f\|^2=(f, f)$ and $|\text{grad}\, f|^2=\sum_{i,j=1}^{2} g^{ij}\partial f/\partial x^i\cdot\overline{\partial f/\partial x^j}$. D_1' is the dual space of D_1 and $L^2(M)$ is imbedded in D_1' in the usual way. We define $\varDelta u$ for $u\in D_1$ as follows; $F\in D_1'$ equals to $\varDelta u$ if

$$F(\phi)=\int u\varDelta\phi dM\tag{2}$$

for any $\phi\in C_0^2(M)$. Let L be the operator with the domain $D(L)=\{f: f\in D_1, \varDelta f\in L^2(M)\}$ and $Lu=\varDelta u$.

Lemma 1. *L is a non-positive definite self-adjoint operator in $L^2(M)$.*