31. *Note on Betti-Numbers of the Module of Differentials*

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Let R be a reduced locality over a perfect field k and let $D_k(R)$ be the module of k-differentials of R (Kähler differentials of R over k). In this note we shall discuss the relations between the Betti-numbers of $D_k(R)$ and some of the algebraic or homological invariants of the k-algebra R.

Throughout this note we assume that all rings are commutative noetherian rings with identity and all modules are finitely generated and unitary.

§1. In this section we shall state notations, definitions and a preliminary lemma. Let A be a ring and M an A-module. We denote by $\dim(A)$ the Krull dimension of A, by $\mathrm{Ass}(M)$ the set of associated prime ideals of M and by $\mathrm{hd}(M)$ the homological dimension of M. In case when A is a local ring with maximal ideal $\mathfrak{m}$, we denote by $r(M)$ the number of minimal generators of M. The number $r(\mathfrak{m})$ is called the embedding dimension of A and denoted by $\mathrm{emdim}(A)$. The dimension of $\mathrm{Tor}_i^A(M, A/\mathfrak{m})$ as a vector space over $A/\mathfrak{m}$ is called the ith Betti-number of M and denoted by $b_i(M)$.

For a local ring A, a composite concept (B, f) of a regular local ring B and a surjective homomorphism $f: B \to A$ is called an embedding of A. An embedding (B, f) of A is said to be minimal if the kernel of f is contained in the square of the maximal ideal of B. It follows from the definition that if (B, f) is a minimal embedding of A, then $\dim(B) = \mathrm{emdim}(A)$.

Let A be a ring and M an A-module. We say that M is torsion free if non zero elements in M are not annihilated by non zero-divisors in A. We shall use later the following:

Lemma. *Let A be a ring and M a torsion free A-module. If $M_\mathfrak{p} = 0$ for all $\mathfrak{p}$ in $\mathrm{Ass}(A)$, then $M = 0$.*

§2. Let R be a locality over a perfect field k, i.e., R is a quotient ring of a finitely generated k-algebra with respect to a prime ideal. Let $D_k(R)$ be the module of k-differentials of R. Let $\mathfrak{m}$ be the maximal ideal of R. We denote by $\mathrm{tr.\,d.}_k(R/\mathfrak{m})$ the transendence degree of the field $R/\mathfrak{m}$ over k. Then we have the following equality:

$$(1) \qquad r(D_k(R_\mathfrak{p})) = \mathrm{emdim}(R_\mathfrak{p}) + \dim(R/\mathfrak{p}) + \mathrm{tr.d.}_k(R/\mathfrak{m})$$

for every prime ideal $\mathfrak{p}$ in R. As a special case of (1), we have