78. *Uniqueness of the Plancherel Measure as an Invariant Measure over the Dual Objects of Compact Groups*

By Nobuhiko TATSUUMA
Department of Mathematics, Kyoto University

(Comm. by Kinjirô KUNUGI, M. J. A., April 12, 1971)

1. In the previous paper [1], we showed an invariancy of the Plancherel measure μ over the dual object $\hat{G}$ of unimodular locally compact groups G of type I under the Kronecker product operations. Especially, for the case that G is compact, the Corollary in [1] gives this invariancy as follows.

Proposition. *For any irreducible (so finite dimensional) unitary representation ω_0, and any function f in $L^1(G) \cap L^2(G)$,*

$$(\dim \omega_0)^{-1}\int_{\hat{G}} |||U_f(\omega_0 \otimes \omega)|||^2 \, d\mu(\omega) = \int_{\hat{G}} |||U_f(\omega)|||^2 \, d\mu(\omega). \tag{1}$$

Here $|||U_f(\omega)|||$ is the Hilbert-Schmidt norm of the operator $U_f(\omega) \equiv \int_G f(g)U_g(\omega)dg$, corresponding to unitary representation $\omega = \{\mathfrak{H}(\omega), U_g(\omega)\}$ of G. (dg is the normalized Haar measure over G as $\int_G dg = 1$.)

In this compact case, as is well known, the integral with respect to μ is just given by the summation with the weight "$\dim \omega$", the dimension of irreducible representation ω, over the discrete dual $\hat{G}$. That is

$$\int_{\hat{G}} \varphi(\omega) d\mu(\omega) = \sum_{\omega \in \hat{G}} \varphi(\omega)(\dim \omega). \tag{2}$$

In the present paper, we shall show that the measure (i.e. weight function) satisfying such a invariancy is unique up to constant factor, that is, the following theorem.

Theorem. *Let $\nu(\omega)$ be a function over the dual $\hat{G}$ of a compact group G.*

If ν satisfies the following equation for any irreducible unitary representation ω_0 of G and for any function f in $L^1(G) \cap L^2(G)$,

$$(\dim \omega_0)^{-1} \sum_{\omega \in \hat{G}} |||U_f(\omega_0 \otimes \omega)|||^2 \nu(\omega) = \sum_{\omega \in \hat{G}} |||U_f(\omega)|||^2 \nu(\omega), \tag{3}$$

then there exists a constant c such that

$$\nu(\omega) = c\,(\dim \omega). \tag{4}$$

Moreover, considering the Plancherel formula for G,

$$\int_G |f(g)|^2 dg = \int_{\hat{G}} |||U_f(\omega)|||^2 d\mu(\omega), \tag{5}$$