121. On the Global Existence of Real Analytic Solutions of Linear Differential Equations. I

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In this note we discuss the problem of global existence of real analytic solutions of linear differential equations $P(D)u=f$ *with constant coefficients*. This problem has remained unsolved by the reason that the topology of the space of real analytic functions on an open set Ω in $\boldsymbol{R}^n$ is complicated (Ehrenpreis [2], Martineau [8]). In fact there has been no general result even when Ω is convex; the only results hitherto known seem to be Theorems α and β below.

The recent theory of sheaf $\mathcal{C}$ (Kashiwara [4], Sato [9]-[11]) however provides us an effective tool on this subject considered thus far to be very difficult, as will be described in this note. The details and complete arguments will be given somewhere else.

Throughout this note we denote by Ω an open set in $\boldsymbol{R}^n$ and by $\bar{\Omega}$ and $\partial\Omega$ its closure and boundary, respectively. $\mathcal{A}$ and $\mathcal{B}$ denote the sheaf of germs of real analytic functions and that of hyperfunctions, respectively. We denote by $\mathcal{A}(\Omega)$ the space of real analytic functions on Ω. For a compact set K in $\boldsymbol{R}^n$ we also denote by $\mathcal{A}(K)$ the space of real analytic functions on K, i.e., $\mathcal{A}(K)=\varinjlim_{V\supset K}\mathcal{O}(V)$, where V is a complex neighbourhood of K and $\mathcal{O}(V)$ denotes the space of holomorphic functions on V. Since we need not know the topological structure of these spaces in our arguments, we do not discuss it here.

We first list up two known theorems for the reader's convenience.

Theorem α (Malgrange [7]). *Let $P(D)$ be elliptic, i.e., have a principal symbol $P_m(\xi)$ never vanishing for any non-zero real cotangent vector ξ. Then for any open set Ω in $\boldsymbol{R}^n$ $P(D)u=f$ has a solution $u(x)$ in $\mathcal{A}(\Omega)$ for any $f(x)$ in $\mathcal{A}(\Omega)$.*

Theorem β (Ehrenpreis). *Let K be a compact convex set in $\boldsymbol{R}^n$. Then $P(D)u=f$ has a solution $u(x)$ in $\mathcal{A}(K)$ for any $f(x)$ in $\mathcal{A}(K)$.*

In the rest of this note we always assume that Ω is a relatively compact domain in $\boldsymbol{R}^n$.

Theorem 1. *Assume that Ω is represented as $\{x\,|\,\varphi(x)<0\}$ by a real valued real analytic function $\varphi(x)$ defined in a neighbourhood of $\bar{\Omega}$ satisfying* $\mathrm{grad}_x\,\varphi\neq 0$ *on $\partial\Omega$. Assume further that $P(D)$ satisfies the following condition* (1) *and Ω satisfies condition* (2). *Then for any*