135. A Note on Distributive Sublattices of a Lattice

By Saburo TAMURA
Yamaguchi University

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In [1], B. Jónsson gave a necessary and sufficient condition for a subset of a modular lattice to generate a distributive lattice. R. Balbes proved Jónsson's theorem without using Zorn's lemma in [2]. In [3], we gave a necessary and sufficient condition that the sublattice generated by a subset H of a lattice should be distributive. In this note we prove this theorem without using Zorn's lemma. And then the condition for the case of $H=\{x, y, z\}$ is expressed by seven lattice polynomial equations.

§1. The finite join of elements in H is called a $\cup$-element. The set of all $\cup$-elements is denoted by $H_\cup$ and dually the set of all $\cap$-elements by $H_\cap$. The finite join of elements in $H_\cap$ is called a $\cup\cap$-element. The set of all $\cup\cap$-elements is denoted by $H_{\cup\cap}$ and dually the set of all $\cap\cup$-elements by $H_{\cap\cup}$.

Two modular laws will be expressed by

μ: $(a\cap c)\cup(b\cap c)=((a\cap c)\cup b)\cap c$, and

μ^*: $(a\cup c)\cap(b\cup c)=((a\cup c)\cap b)\cup c$.

Four distributive laws will be expressed by

δ: $(a\cap c)\cup(b\cap c)=(a\cup b)\cap c$,

δ^*: $(a\cup c)\cap(b\cup c)=(a\cap b)\cup c$,

Δ: $\bigcup_{i=1}^m (x_i\cap y)=(\bigcup_{i=1}^m x_i)\cap y$, and

Δ^*: $\bigcap_{i=1}^m (x_i\cup y)=(\bigcap_{i=1}^m x_i)\cup y$.

Theorem 1. *Let H be a nonempty subset of a lattice L. In order for the sublattice of L generated by H to be distributive, it is necessary and sufficient that*

Δ holds for any $x_1, \cdots, x_m \in H$ and any $y\in H_\cap$,

μ holds for any $a\in H_\cap$ and any $b, c\in H_{\cup\cap}$, and

μ^* holds for any $b\in H_\cap$ and any $a, c\in H_{\cup\cap}$.

Proof. The modular laws used in the proof of [2] are only those laws mentioned above.

Corollary 2. *Let $\langle H\rangle$ be the sublattice generated by a nonempty subset H of a lattice. The following four statements are equivalent.*

(i) *$\langle H\rangle$ is distributive.*

(ii) *δ holds for any $a, b, c\in H_{\cup\cap}$.*

(iii) *Δ holds for any $x_1, \cdots, x_m\in H$ and any $y\in H_\cap$, and μ^* holds for any $a, b, c\in H_{\cup\cap}$.*