2. A Note on Primitive Extensions of Rank 4 of Alternating Groups

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1. In [11] and [5], the primitive extensions of rank 3 of symmetric groups or alternating groups which act naturally have been determined. More generally, E. Bannai has determined the primitive extensions of rank 3 of 4-ply transitive permutation groups ([1]). In this note we consider the primitive extensions of rank 4 of alternating groups which act naturally. Before stating our result, we note the following. It gives an example of primitive and imprimitive groups of an arbitrary rank.

Let H_k be the symmetric group S_k or the alternating group A_k which act naturally on $\{1, 2, \cdots, k\}$ and let E_{k-1} be the elementary abelian group of order 2^{k-1}. Furthermore, let $a_1, a_2, \cdots, a_{k-1}$ be a minimal set of generators of E_{k-1} and put $a_k = a_1 a_2 \cdots a_{k-1}$. Every element σ of H_k induces an automorphism $\bar{\sigma}$ of E_{k-1} defined by

$$\bar{\sigma} = \begin{pmatrix} a_1\ a_2\ \cdots a_k\ a_1a_2\ \ \cdots \\ a_{1\sigma}a_{2\sigma} \cdots a_{k\sigma}a_{1\sigma}a_{2\sigma} \cdots \end{pmatrix}.$$

Thus H_k is identified with an automorphism group of E_{k-1}.
Construct the semidirect product $H_k \cdot E_{k-1}$ and let it act naturally on $\{H_k x \mid x \in E_{k-1}\}$, the set of right cosets of H_k. Then we have

Proposition (cf. 4 (iv) in Tsuzuku [11]). *For $n \geqq 2$, $H_{2n} \cdot E_{2n-1}$ is an imprimitive rank $n+1$ group of degree 2^{2n-1} with subdegrees $1, \binom{2n}{1}, \binom{2n}{2}, \cdots, \binom{2n}{n-1}, 1/2\binom{2n}{n}$, and $H_{2n+1} \cdot E_{2n}$ is a primitive rank $n+1$ group of degree 2^{2n} with subdegrees*

$$1, \binom{2n+1}{1}, \binom{2n+1}{2}, \cdots, \binom{2n+1}{n-1}, \binom{2n+1}{n}.$$

Theorem. *Let A_k be the alternating group of degree k. If A_k has a primitive extension G of rank 4, then $k=7$ and G is isomorphic to $A_7 \cdot E_6$.*

2. Outline of a proof of Theorem.

Notation.

S_k: The symmetric group of degree k.

A_k: The alternating group of degree k (on a set Δ_1).

G: A primitive extension of rank 4 of A_k on a set $\Omega = \{0, 1, 2, \cdots, k, \tilde{1}, \tilde{2}, \cdots, \tilde{l}, \bar{1}, \bar{2}, \cdots, \bar{m}\}$.