54. *Necessary and Sufficient Conditions for Countable Compactness of Product Spaces*

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In this paper a product space is the topological product of two spaces.

We know in [1] the necessary and sufficient conditions for normality of product spaces. The conditions are obtained by describing the normality of product space on its factor spaces. The idea of such the description has been used explicitly or implicitly in some literatures. In fact, the description way is useful and sometimes necessary in the study of product spaces as [2], [3] and [4] also show.

In this paper we intend to present another application of the description method by giving the conditions stated in the title. Since 1953 when Terasaka [13] and Novák [11], answering a question posed by Čech in 1938, showed that the product of two countably compact spaces need not be countably compact, the condition for countable compactness of a product space has been searched and several sufficient ones were obtained. We shall now show complete conditions in Theorem 2. One of merits of the theorem will be seen in Corollary 6 to the theorem which generalizes a theorem of Frolík [8] and is proved fairly more simply than in [8]. Some topics on the closedness of projection maps of product spaces are also discussed.

Throughout this paper, unless otherwise stated, spaces are T_1 spaces.

We first recall some notations and results in [1] and [2] for later use. For a subset A of the product space $X\times Y$ we write $A[x]=\{y\,;(x,y)\in A\}$ for each point $x\in X$. Let $\mathfrak{F}=\{F_x\,;x\in Z\subset X\}$ be a family of subsets of Y with suffixes of points in X, then we write

$$\limsup_a \mathfrak{F}=\limsup_a F_x=\bigcap_{U\in\mathfrak{N}_a}\overline{\bigcup_{x\in U}F_x}$$

for any point $a\in X$, where $\mathfrak{N}_a$ is the neighbourhood system of a in X. We know [1] that, putting $F=\{(x,y)\,;x\in Z,\,y\in F_x\}\subset X\times Y$, we have

$$(*)\qquad \bar{F}[a]=\limsup_a F_x$$

for any $a\in X$; and that

$$(**)\qquad \limsup_a F_x=\limsup_a(\limsup_z F_x).$$

A space Y is *upper compact at a point* a of X if for any family $\mathfrak{F}=\{F_x\,;x\in Z\subset X\}$ of non-void subsets of Y with $a\in\bar{Z}-Z$ we have