51. A Note on the Dilation Theorems

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1. Introduction. In the recent decade, the so-called harmonic analysis of operators grew rapidly by the works mainly due to Sz. Nagy's school, cf. [5]. The main tool in their investigations is the following strong dilation theorem due to Sz. Nagy:

Theorem A. *If T is a contraction acting on a Hilbert space $\mathfrak{H}$, then there is a unitary U acting on a Hilbert space $\mathfrak{K}$ including $\mathfrak{H}$ as a subspace such that*

(1) $$T^n = PU^n|\mathfrak{H} \qquad (n=0,1,2,\cdots),$$

where P is the projection of $\mathfrak{K}$ onto $\mathfrak{H}$.

By the importance of the theorem, several proofs are given, cf. [5; Chapter I]. Some of them are based on the following general dilation theorems due to Naimark, cf. [3], [5].

Theorem B. *If $F(A)$ is a positive operator-valued measure defined on a σ-field $\mathfrak{B}$ of sets and $F(A)$ acts on $\mathfrak{H}$, then there is a spectral measure $E(A)$ of $\mathfrak{B}$ acting on $\mathfrak{K}$ including $\mathfrak{H}$ such that*

(2) $$F(A) = PE(A)|\mathfrak{H} \qquad (A \in \mathfrak{B}).$$

Theorem C. *If $V(g)$ is an operator-valued positive definite function defined on a group G and $V(g)$ acts on $\mathfrak{H}$, then there is a unitary representation $U(g)$ of G on $\mathfrak{K}$ including $\mathfrak{H}$ such that*

(3) $$V(g) = PU(g)|\mathfrak{H} \qquad (g \in G).$$

However, there is an another general dilation theorem due to Stinespring [4] and Umegaki [6] which receives less attentions:

Theorem D. *If $V(a)$ is a completely positive (or positive definite in the sense of [6]) linear mapping of a $*$-algebra $\mathcal{A}$ into $\mathcal{B}(\mathfrak{H})$, the algebra of all (bounded linear) operators acting on $\mathfrak{H}$, then there is a $*$-homomorphism $\Phi(a)$ of $\mathcal{A}$ into $\mathcal{B}(\mathfrak{K})$ where $\mathfrak{K}$ includes $\mathfrak{H}$ and Φ satisfies*

(4) $$V(a) = P\Phi(a)|\mathfrak{H} \qquad (a \in \mathcal{A}).$$

It seems to the authors that there is no literature which gives a proof that Theorem D implies Theorem A. In § 2, we shall give some theorems proofs.

Umegaki [6] pointed out that Theorem C implies Theorem D if $\mathcal{A}$ is the group algebra of a locally compact group G. The converse of this implication obviously follows from