82. On Representations of Homology Classes

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1. Introduction. R. Thom [3] has shown that every integral $(n-1)$-dimensional homology class θ of an orientable n-manifold M is representable by an $(n-1)$-submanifold of M. In this result the submanifold representing θ is not required to be connected. In the present paper, we shall consider under what condition θ is representable by a connected $(n-1)$-submanifold. Our result is stated as follows:

Theorem. *Let M be a compact connected orientable manifold of dimension $n\geq 3$ with connected boundary (possibly empty). Let $\{g_1, \cdots, g_r\}$ be a free basis for the group $H_{n-1}(M;Z)$. Then, for a non-zero homology class*

$$\theta=a_1g_1+\cdots+a_rg_r(a_i\in Z),$$

the following conditions are mutually equivalent;

(i) *θ can be represented by a connected $(n-1)$-submanifold.*

(ii) *The greatest common devisor $(|a_1|, \cdots, |a_r|)$ is 1.*

(iii) *There is a homology class $\alpha\in H_1(M;Z)$ such that the intersection $\theta\cdot\alpha$ is 1.*

Everything will be considered from the *PL* viewpoint. However we note that the similar argument is applicable in the differentiable viewpoint. I am grateful to Mr. K. Yokoyama for his suggestions given me at the very beginning of this work.

2. Attaching handles. Throughout this paper all manifolds, with or without boundary, are to be compact, oriented and *PL*. All submanifolds of a manifold M are, moreover, to be closed and locally flat in M. The boundary of a manifold M is denoted by ∂M and the interior of M by int M. The manifold M with orientation reversed is denoted by $-M$.

Let A be an $(n-1)$-submanifold of an n-manifold M and let $f:I=[0,1]\to M$ be a simple arc. When $f(I)$ meets A transversely at $P\in M$, the intersection number of A and f at P, denoted by sign $(A,f:P)$, is defined as follows: Since f is transversal to A at P, there exists a *PL* homeomorphism $h:U\to B^{n-1}\times B^1$ so that $h(P)=(0,0)$, $h(U\cap A)=B^{n-1}\times 0$ and $h(U\cap f(I))\subset 0\times B^1$, where U is a ball neighborhood of P in M and B^i denotes a *PL* i-ball and 0 denotes an interior point corresponding to the barycenter of the standard simplex. Choose h so that $U\cap A$, $U\cap f(I)$ are mapped with natural orientation. We then define