# 107. A Note on Cogenerators in the Category of Modules

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Let $A$ be a ring with identity and ${}_AW$ a cogenerator in the category of unitary left $A$-modules, and denote by $B=\mathrm{End}\,({}_AW)$ the endomorphism ring of ${}_AW$. Then $W$ is regarded as an $A$-$B$-bimodule. As for the structure of ${}_AW$ in general, there was a useful result of Osofsky [5, Lemma 1]. As for the structure of $W_B$, recently Onodera has obtained an interesting result [4, Theorem 1].

The purpose of this paper is to establish the following two theorems:

**Theorem 1.** *Let ${}_AW$ be a cogenerator, and let $B=\mathrm{End}\,({}_AW)$ and $C=\mathrm{End}\,(W_B)$. Then $W_B$ is absolutely pure and semi-injective. Furthermore $A$ is dense in $C$ relative to the finite topology. In particular, if ${}_AW$ is finitely cogenerating in the sense of Morita* [3], *then ${}_AW$ possesses the double centralizer property, i.e. $C=A$.*

**Theorem 2.** *Let ${}_AW$ be a cogenerator and $B=\mathrm{End}\,({}_AW)$, and denote by $S(W_B)$ the socle of $W_B$. Let further $\{V_\lambda \mid \lambda \in \Lambda\}$ be a complete representative system of isomorphism classes of simple left $A$-modules such that $E(V_\lambda)\subset W$ for each $\lambda\in\Lambda$ (Cf.* [5, *Lemma* 1]), *where $E(V_\lambda)$ denotes an injective hull of $V_\lambda$. Then $S(W_B)\subset' W_B$, and*

$$S(W_B)=\sum_{\lambda\in\Lambda}\oplus V_\lambda B$$

*is the decomposition of $S(W_B)$ into homogeneous components.*

Throughout this paper, all modules are assumed to be unitary, and we shall keep above notations and meanings. In particular, ${}_AW$ denotes always a cogenerator and $B$ (resp. $C$) denotes the endomorphism ring of ${}_AW$ (resp. of $W_B$).

## 1. Proof of Theorem 1.

Previous to this, we need some lemmas.

**Lemma 1** [4, Theorem 1]. *Let $M$ be a left $A$-module and set $M_B^*=\mathrm{Hom}_A({}_AM, {}_AW_B)$. Then, for each finitely generated $B$-submodule $U$ of $M_B^*$ and for each $B$-homomorphism $f: U_B\to W_B$, there exists an element $v$ in $M$ such that $f=\rho(v)\cdot i$, where $i: U_B\to M_B^*$ implies the inclusion map and $\rho: M\to \mathrm{Hom}_B(M_B^*, W_B)$ is the canonical map defined by $\rho(x)(g)=g(x)$ for every $x\in M$ and $g\in M^*$.*

Let us denote by $W^n$ (resp. $B^n$) the direct sum of $n$ copies of $W$ (resp. of $B$). For a subset $X$ of $W^n$, set

$$(0:X)_{B^n}=\{(b_1,\cdots,b_n)\in B^n \mid \textstyle\sum v_ib_i=0 \quad \text{for all } (v_1,\cdots,v_n)\in X\}.$$

Similarly for a subset $Y$ of $B^n$, set