106. *Modified Korteweg - de Vries Equation and Scattering Theory*

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1. Introduction. Gardner, Greene, Kruskal and Miura (G. G. K. M.) [1] have discovered that the initial value problem for the Korteweg - de Vries (KdV) equation

$$v_t + 6vv_x + v_{xxx} = 0$$

(subscripts x and t denoting partial differentiations) may be exactly solved by the direct and inverse scattering theory of the one dimensional Schrödinger operator. Zakharov and Shabat [9] have then developed an analogue of G. G. K. M. theory for the non-linear Schrödinger equation

$$iu_t + 2^{-1}u_{xx} + |u|^2 u = 0 \tag{1}$$

relating it to the scattering theory of the differential operator

$$L_u = i\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}D - i\begin{bmatrix}0 & u\\ u^* & 0\end{bmatrix} \qquad D = d/dx$$

with complex potential u (u^* being its complex conjugate).

Recently Wadati [8] and the present author [7] have noted that the modified KdV equation

$$v_t + 6v^2 v_x + v_{xxx} = 0 \tag{2}$$

(v being real-valued) can be also related to the operator L_u. In [7] a family of particular solutions of (2) have been explicitly constructed based on this relation. In this paper we supplement [7] with the description of more general aspect of the theory.

2. Evolution equations for linear operators. Lax [3], [4] has rewritten the KdV equation into the evolution equation for the Scröd-inger operator. An analogous result also holds for equation (2): Put

$$A_v = -4D^3 + 3\begin{bmatrix}-v^2 & iv_x\\ iv_x & -v^2\end{bmatrix}D + 3D\begin{bmatrix}-v^2 & iv_x\\ iv_x & -v^2\end{bmatrix}$$

where v is a real valued function. Then by direct calculation, the equation (2) is rewritten into the form

$$(L_{iv})_t = [A_v, L_{iv}] = A_v L_{iv} - L_{iv} A_v. \tag{3}$$

This expression has been obtained in [7].

Remark 1. Put

$$B_u = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}(D^2 + 2^{-1}|u|^2) - 2^{-1}\begin{bmatrix}0 & u\\ u^* & 0\end{bmatrix}D - 2^{-1}D\begin{bmatrix}0 & u\\ u^* & 0\end{bmatrix}$$