105. *A General Local Ergodic Theorem*

By Yoshihiro KUBOKAWA
The Institute of Statistical Mathematics

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1. Introduction and the theorem. The purpose of this note is to prove a local ergodic theorem for a one-parameter semi-group of positive bounded linear operators on $L_1(X)$. A local ergodic theorem for a one-parameter semi-group of positive linear contractions was proved by Krengel [5], Ornstein [6], Akcoglu-Chacon [1] and Terrell [7] under little different conditions. Fong-Sucheston gave a proof of a local ergodic theorem for a special class of one-parameter semi-groups of positive uniformly bounded linear operators [4].

Let $(X, \mathfrak{B}, m)$ be a σ-finite measure space and $L_1(X)=L_1(X, \mathfrak{B}, m)$ be the Banach space of real integrable functions on X. Let $(T_t)(t\geqslant 0)$ be **a strongly continuous one-parameter semi-group of positive bounded linear operators on $L_1(X)$.** This means that ① T_t is a positive bounded linear operator on $L_1(X)$ for every $t\geqslant 0$ and $T_0=I$ (identity) (The positivity of T means that $Tf\geqslant 0$, if $f\geqslant 0$.), ② $T_{t+s}f=T_t\circ T_sf$ for any t, $s\geqslant 0$ and $f\in L_1(X)$, ③ $\lim_{t\to 0}\|T_tf-f\|=0$ for any $f\in L_1(X)$ (strong continuity). Then there exist constants M, β such that $\|T_t\|\leqslant Me^{\beta t}$ [9]. (If we can take $M=1$, $\beta=0$, then (T_t) is said to be a strongly continuous one-parameter semi-group of positive linear contractions.) By the strong continuity of (T_t), there exists a function $g(t, x)$ such that for a fixed $t\geqslant 0$, $g(t, x)=(T_tf)(x)$ for a.e. x and $g(t, x)$ is $\mathfrak{L}^+\times\mathfrak{B}$-measurable, where $\mathfrak{L}^+$ is the σ-algebra of Lebesgue measurable sets on the half real line $[0, \infty)$. The function with this property is uniquely determined in the class of $\mathfrak{L}^+\times\mathfrak{B}$-measurable functions [3, 8]. We define the integral $\int_a^b (T_tf)(x)dt$ $(0\leqslant a<b<\infty)$ by $\int_a^b g(t, x)dt$.

We shall prove the following

Theorem. *Let (T_t) be a strongly continuous one-parameter semi-group of positive bounded linear operators on $L_1(X)$. Then we have*

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha (T_tf)(x)dt=f(x) \qquad a.e. \text{ for any } f\in L_1(X).$$

Corollary. *If $g\geqslant 0$ and $g\in L_1(X)$, then we have*

$$\lim_{\alpha\to 0}\frac{\int_0^\alpha (T_tf)(x)\,dt}{\int_0^\alpha (T_tg)(x)\,dt}=\frac{f(x)}{g(x)} \qquad a.e. \text{ for any } f\in L_1(X)$$