# 131. On Certain Identities between the Traces of Hecke Operators

By Hiroaki HIJIKATA

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Linear relations between the traces of Hecke operators and those of the Brandt matrices were first obtained by Eichler [1] and [2], then generalized by Shimizu [6]. In this note, we shall further generalize (with respect to the levels of the groups involved) and in a sense sharpen (by restricting the operators to the essential parts) their results.

Let $\boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}$ and $\boldsymbol{C}$ denote the set of integers, rational numbers, real numbers and complex numbers respectively. For a prime $p$, let $\boldsymbol{Z}_p$ and $\boldsymbol{Q}_p$ denote the set of $p$-adic integers and $p$-adic numbers. For a ring $M$ with a unity, let $M^\times$ denote the group of invertible elements of $M$.

Let $B$ be a quaternion algebra having $\boldsymbol{Q}$ as its center. Let $d^2$ be its discriminant, i.e. $d$ is a product of all distinct primes $p$ where the completion $B_p = B \otimes \boldsymbol{Q}_p$ is a division algebra. We admit the case where $d=1$, namely $B$ is the two by two total matrix algebra $M_2(\boldsymbol{Q})$. Let $N$ be a product of $d$ and a natural number $M$ prime to $d$, $N=dM$, $(d, M)=1$. An order $\boldsymbol{I}$ of $B$ is called a *split order of level* $N$, if it satisfies (i) if $p \mid d$, $\boldsymbol{I}_p = \boldsymbol{I} \otimes \boldsymbol{Z}_p$ is a maximal order of $B_p$, and (ii) if $(p, d)=1$, there is an isomorphism $\varphi_p : B_p \to M_2(\boldsymbol{Q}_p)$ over $\boldsymbol{Q}_p$ such that $\varphi_p(\boldsymbol{I}_p) = \begin{pmatrix} \boldsymbol{Z}_p & \boldsymbol{Z}_p \\ M\boldsymbol{Z}_p & \boldsymbol{Z}_p \end{pmatrix}$. Now we fix a split order $\boldsymbol{I}$ of level $N$, and an isomorphism $\varphi_p : B_p \to M_2(\boldsymbol{Q}_p)$ for each $p$ prime to $d$, and write $\varphi_p(x) = \begin{pmatrix} a_p(x) & b_p(x) \\ c_p(x) & d_p(x) \end{pmatrix}$ for $x \in B_p$. In the following assume that $B$ is indefinite unless otherwise stated, and fix an isomorphism $\varphi : B \otimes \boldsymbol{R} \to M_2(\boldsymbol{R})$. Let $\boldsymbol{I}^1$ denote the group of all elements of reduced norm 1 in $\boldsymbol{I}$. Let $\Gamma = \Gamma(\boldsymbol{I}) = \varphi(\boldsymbol{I}^1)$, and we identify $\Gamma$ with $\boldsymbol{I}^1$, when convenient. Then $\Gamma$ is a subgroup of the connected component $GL_2^+(\boldsymbol{R})$ of $GL_2(\boldsymbol{R})$. The group $GL_2^+(\boldsymbol{R})$ is acting on the complex upper half plane $\boldsymbol{H}$ as linear fractional transformations. Under this action, $\boldsymbol{H}/\Gamma$ has a finite invariant volume, and it is compact if and only if $d>1$.

Let $c$ be a divisor of $M$, and $\chi : (\boldsymbol{Z}/c\boldsymbol{Z})^\times \to \boldsymbol{C}^\times$ be a primitive character modulo $c$. Let $\Delta$ be the subset of $\boldsymbol{I}$ consisting of all elements $x$ such that $a_p(x) \not\equiv 0 \bmod p$ for any prime $p$ dividing $M$. Starting from the given character $\chi$, let us define the map $\chi : \Delta \to \boldsymbol{C}^\times$ by the formula $\chi(x) = \prod_{p \mid c} \chi(a_p(x))$ for $x \in \Delta$. This new $\chi$ is multiplicative on $\Delta$, and its