145. *Analogue of Fourier's Method for Korteweg - de Vries Equation*

By Shunichi TANAKA
Department of Mathematics, Osaka University

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1. Introduction. In this paper we study the Korteweg - de Vries (KdV) equation

(1) $u_t - 6uu_x + u_{xxx} = 0 \qquad u = u(t) = u(x, t) - \infty < x, t < \infty$

for rapidly decreasing initial data. Gardner, Greene, Kruskal and Miura (G.G.K.M.) [2] have associated one dimensional Schrödinger operators $L_{u(t)} = -(d/dx)^2 + u(t)$ to a solution of (1). They have found a simple formula describing the time variation of scattering data of $L_{u(t)}$. This paper is concerned with converse statement which may be viewed as a non-linear analogue of Fourier's method for solving linear partial differential equations of mathematical physics: Given the initial value one determine the scattering data of $L_{u(0)}$. Define scattering data for each t according to the formula of G.G.K.M. Using inverse scattering theory, one can construct potential $u(x, t)$ with prescribed scattering data for each t. Then $u(x, t)$ is a solution of (1).

Throughout the paper subscripts with independent variables denote partial differentiations. Integrations are taken over $(-\infty, \infty)$ unless explicitly indicated.

2. Preparation from scattering theory. Consider one dimensional Schrödinger equation

(2) $$-\phi_{xx} + u(x)\phi = \zeta^2\phi.$$

Under the assumption that $(1 + |x|)u(x)$ is integrable, the inverse scattering theory for (2) has been solved by Marchenko for the half line $(0, \infty)$ and then the case of the infinite interval has been treated by Faddeev [1]. We follow [1] in this paper.

For each $\zeta = \xi + i\eta, \eta \geq 0$, there exist unique solutions $f_\pm(x, \zeta)$ which behave like $\exp(\pm i\zeta x)$ as $x \to \pm\infty$. They are called Jost solutions of (2). Jost solutions are analytic in ζ, $\mathrm{Im}\,\zeta > 0$. If $\zeta = \xi$ non-zero real, then f_+ and its complex conjugate f_+^* are independent solutions of (2). One can express f_- as $f_- = a(\xi) f_+^* + b(\xi) f_+$. $a(\xi)$ is limiting value of a function $a(\zeta)$ analytic in ζ, $\mathrm{Im}\,\zeta > 0$. The (right) reflection coefficient $r(\xi) = b(\xi)a(\xi)^{-1}$ is defined for $\xi \neq 0$ and its absolute value is bounded by 1. $a(\zeta)$ has only a finite numbers of zeros. They are all simple and purely imaginary. We denote them by $i\eta_1, \cdots, i\eta_N$. $f_\pm$ are linearly dependent for $\zeta = i\eta_j$ and are square integrable because of the asymptotic