1. *A Note on the Selberg Sieve and the Large Sieve*

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(Comm. by Kunihiko KODAIRA, M. J. A., Jan. 12, 1973)

1. Let $X \geq 1$ be a real number, and let M, N be integers with $N>0$. Suppose that there are $\omega(p)$ residue classes $a_{p,j} \pmod p$ $(j=1, \cdots, \omega(p)$, $0 \leq \omega(p) < p)$ corresponding to each prime p, and put $P=\prod_{p\leq X} p$. We define then the integer sequence f_n by

$$f_n=\sum_{p\leq X} Pp^{-1}\prod_{j=1}^{\omega(p)}(n-a_{p,j}), \qquad (n=M+1, \cdots, M+N).$$

A. Selberg's sieve method to estimate the quantity

$$Z=\sum_{\substack{n=M+1\\(f_n,P)=1}}^{M+N} 1$$

from above is as follows: define multiplicative arithmetic functions ω, Φ, ρ by

$$\omega(d)=\begin{cases}\prod_{p\mid d}\omega(p), & \text{if } d\mid P,\\ 0, & \text{otherwise,}\end{cases}$$

$$\Phi(q)=q\prod_{p\mid q}\left(1-\frac{\omega(p)}{p}\right),$$

$$\rho(q)=\mu^2(q)\prod_{p\mid q}\frac{\omega(p)}{p-\omega(p)},$$

for natural numbers d, q, and put

$$\lambda_d=\mu(d)\frac{d}{\Phi(d)}\left(\sum_{q\leq X}\rho(q)\right)^{-1}\left(\sum_{\substack{q\leq X/d\\(q,d)=1}}\rho(q)\right),$$

$$R(d)=\sum_{\substack{n=M+1\\ d\mid f_n}}^{M+N} 1-\frac{\omega(d)}{d}N.$$

Then we have

$$Z\leq\sum_{n=M+1}^{M+N}\left(\sum_{d\mid f_n}\lambda_d\right)^2=\frac{N}{\sum_{q\leq X}\rho(q)}+\sum_{d_1\leq X}\sum_{d_2\leq X}\lambda_{d_1}\lambda_{d_2}R([d_1, d_2]), \tag{1.1}$$

where $[d_1, d_2]$ denotes the least common multiple of d_1, d_2, [5].

On the other hand the large sieve method [1], [4] gives

$$\left|\sum_{\substack{n=M+1\\(f_n,P)=1}}^{M+N} a_n\right|^2\sum_{q\leq X}\rho(q)\leq(N+2X^2)\sum_{\substack{n=M+1\\(f_n,P)=1}}^{M+N}|a_n|^2$$

for arbitrary complex numbers a_n $(n=M+1, \cdots, M+N)$, so that we have

$$Z\leq\frac{N+2X^2}{\sum_{q\leq X}\rho(q)}, \tag{1.2}$$