93. *Amenable Transformation Groups. II*

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Introduction. Let X be a nonvoid set and G be a group of transformations of X onto itself. Then we shall say the pair $X=(G,X)$ is a *transformation group*. Let $m(X)$ be the Banach space of all bounded real functions on X and $m(X)^*$ the conjugate Banach space of $m(X)$. For every $s\in G$, define the mapping $l_s: m(X)\to m(X)$ by $l_s f={}_s f$ for any $f\in m(X)$ where ${}_s f(x)=f(sx)$ for $x\in X$, and denote by L_s the adjoint of l_s. For $\varphi\in m(X)^*$ it is called a *mean* if $\varphi\geqq 0$ and $\varphi(I_X)=1$ where I_X is the constant one function on X. A mean φ is called *multiplicative* if $\varphi(f\cdot g)=\varphi(f)\cdot\varphi(g)$ for any $f,g\in m(X)$. For a subset K of G, a mean φ is *K-invariant* if $L_s\varphi=\varphi$ for all $s\in K$. We denote by δ_x the Dirac measure at $x\in X$. Let $IM(X)$ [βX] be the set of all G-invariant [multiplicative] means. We shall say the transformation group $X=(G,X)$ is *amenable* if $IM(X)$ is nonempty.

The purpose of this paper is to characterize the transformation group $X=(G,X)$ such that $IM(X)\cap Co(\beta X)$ is nonempty where $Co(\beta X)$ is the convex hull of βX and to study the extreme point of the convex set $IM(X)\cap Co(\beta X)$. For semigroups the analogous problem is investigated by A. T. Lau in [3] and [4].

§1. Multiplicative means. In this section we give the Lemmas used in later sections. Let $X=(G,X)$ be a transformation group and $\varphi\in m(X)^*$ be a mean. For any subset A of X, we write $\varphi(A)$ instead of $\varphi(I_A)$ where I_A is the characteristic function of A. We put $H(\varphi)=\{s\in G: L_s\varphi=\varphi\}$.

Lemma 1. *Let* $\Phi=\{\varphi_i\in\beta X: i=1,2,\cdots,m \text{ and } \varphi_i\neq\varphi_j \text{ if } i\neq j\}$ *and* $\Psi=\{\psi_i\in\beta X: i=1,2,\cdots,n \text{ and } \psi_i\neq\psi_j \text{ if } i\neq j\}$. *If* $\sum_{i=1}^m\lambda_i\varphi_i=\sum_{i=1}^n\mu_i\psi_i$ *where* λ_i*'s and* μ_i*'s are positive numbers, then* $\Phi=\Psi$.

Lemma 2. *Let* $\varphi_0\in\beta X$. *For a subset* $\{a_1,a_2,\cdots,a_n\}$ *of* G *put* $\varphi_i=L_{a_i}\varphi_0\in\beta X$ *for* $1\leqq i\leqq n$. *If* $\varphi_1,\varphi_2,\cdots,\varphi_n$ *are mutually distinct, there is a subset* $A_0\subset X$ *such that for any* $1\leqq i,j\leqq n$ $\varphi_i(A_j)=\delta_{ij}$ *and* $A_i\cap A_j=\phi$ $(i\neq j)$ *where* $A_i=a_iA_0$.

Now for a mean φ we consider the condition (♯): *there is a positive constant c such that* $\varphi(A)\geqq c$ *or* $\varphi(A)=0$ *for any* $A\subset X$. If the condition (♯) is satisfied, there is a subset $A\subset X$ such that $\varphi(A)>0$ and that $\varphi(A\cap B)$ is equal to $\varphi(A)$ or 0 for any $B\subset X$. For example, every $\varphi\in Co(\beta X)$ satisfies the condition (♯).

Lemma 3. *Let* $\varphi\in IM(X)$ *satisfy the condition* (♯) *and* A *be a*