39. A Classification of Compact 3-Manifolds with Infinite Cyclic Fundamental Groups

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I. Results. We consider a compact connected piecewise linear 3-manifold M^3 which may be either orientable or non-orientable. If there is a component of the boundary ∂M^3 of M^3 which is homeomorphic to S^2, we attach a 3-cell to eliminate it. Note that the orientability of the resulting manifold coincides with that of the original one. Thus we assume that the boundary ∂M^3 contains no components which are homeomorphic to S^2 throughout this note. Under this assumption compact 3-manifolds with $\pi_1=Z, Z$ being an infinite cyclic group will be classified modulo Poincaré Conjecture. The classification implies that such a manifold is essentially the S^2-bundle over S^1: $S^1\times S^2$, the twist S^2-bundle over S^1: $S^1\times_\tau S^2$, the solid torus: $S^1\times B^2$ or the solid Klein bottle: $S^1\times_\tau B^2$.

First, by using results of H. Kneser [2], J. H. C. Whitehead [8] and J. W. Milnor [3], we shall prove the following:

Theorem 1. *If $\partial M^3=\phi$ and $\pi_1(M^3)=Z$ then M^3 is homeomorphic to the connected sum $(S^1\times S^2)\#\tilde{S}^3$ or $(S^1\times_\tau S^2)\#\tilde{S}^3$ according as M^3 is orientable or non-orientable, where $\tilde{S}^3$ is a homotopy 3-sphere.*

Next, using Partial Poincaré Duality due to the present author [1], we shall obtain the following:

Theorem 2. *If $\partial M^3\neq\phi$ and $\pi_1(M^3)=Z$ then M^3 is homeomorphic to $(S^1\times B^2)\#\tilde{S}^3$ or $(S^1\times_\tau B^2)\#\tilde{S}^3$ according as M^3 is orientable or non-orientable. In particular, in case M^3 is orientable, M^3 may be considered as $cl(\tilde{S}^3$-unknotted solid torus).*

From Theorems 1 and 2 we obtain the following Conclusion:

Conclusion. *Any compact connected 3-manifold with $\pi_1=Z$ is homeomorphic to $(S^1\times S^2)\#\tilde{S}^3$, $(S^1\times_\tau S^2)\#\tilde{S}^3$, $(S^1\times B^2)\#\tilde{S}^3$ or $(S^1\times_\tau B^2)\#\tilde{S}^3$ with a finite number of open 3-cells removed.*

II. Sketch of proofs. Proofs will be considered in the piecewise linear category.

Proof of Theorem 1. By a result of H. Kneser [2], M^3 is homeomorphic to $P\#\tilde{S}^3$, where P is a prime 3-manifold in the sense that if P is homeomorphic to $P_1\#P_2$ then P_1 or P_2 is a 3-sphere. Since $\pi_1(P)=Z$, from the sphere theorem in the sense of J. H. C. Whitehead [8], we