75. On Hodge Structure of Isolated Singularity of Complex Hypersurface

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Introduction. The Hodge spectral sequence for an isolated singularity of (complex) analytic space is defined as follows. Note first that, given a complex manifold Z, the bigrading of differential forms of Z together with the operators ∂ and $\bar{\partial}$ defines a double complex. The Hodge structure $(E_r^{p,q}(Z), d_r)$ of Z is the spectral sequence of this double complex so chosen that $E_1^{p,q}(Z)=H^q(Z,\Omega_Z^p)$ where Ω_Z^p denotes the sheaf of holomorphic p-forms on Z. Let now (X,x) denote the situation where x is an isolated singular point of an analytic space X. For sufficiently small neighborhood U of x, $(E_r^{p,q}(U\backslash x), d_r)$ are well defined and form a direct system with the restriction maps. Set

$$E_r^{p,q}(X,x)=\varinjlim_{U} E_r^{p,q}(U\backslash x).$$

The map $d_r: E_r^{p,q}(X,x)\to E_r^{p+r,q-r+1}(X,x)$ is naturally induced. $(E_r^{p,q}(X,x), d_r)$ thus obtained is the Hodge spectral sequence of the isolated singularity (X,x). If X is n-dimensional, then $E_r^{p,n}(X,x)=0$ by Malgrange [3]. By Andreotti-Grauert [1] $E_1^{p,q}(X,x)$ are finite-dimensional (over $\boldsymbol{C}$) if $1\leqq q\leqq n-2$.

The main result is the following

Theorem 1. *Let $n\geqq 3$ and suppose (X,x) is a hypersurface singularity, that is, there is a holomorphic function f in a domain Y of $\boldsymbol{C}^{n+1}: (z_0,\cdots,z_n)$ such that $X=\{z\in Y; f(z)=f(x)\}$, and such that $\partial f(z)/\partial z_i=0$ $(0\leqq i\leqq n)$ if and only if $z=x$. Let $E_r^{p,q}(X,x)$ be denoted for short by $E_r^{p,q}$. Then the following conclusions are valid.*

(i) $E_1^{p,q}=0$ *if* $q\neq 0$, $q\neq n-1$, $p+q\neq n-1$, $p+q\neq n$.

(ii) *There are canonical isomorphisms:*

$$E_1^{2,n-2}\cong E_1^{3,n-3}\cong\cdots\cong E_1^{n-1,1}$$
$$E_1^{1,n-2}\cong E_1^{2,n-3}\cong\cdots\cong E_1^{n-2,1}$$

(ii)′ $\dim E_1^{n-q,q-1}=\dim E_1^{n-q,q}$ *for* $2\leqq q\leqq n-2$

(iii) $E_2^{p,q}$ *are all finite-dimensional.*

(iv) $E_2^{p,0}=0$ *for* $1\leqq p\leqq n-2$.

(iv)′ $E_2^{p,n-1}=0$ *for* $2\leqq p\leqq n-1$.

(v) *If μ is the multiplicity of the hypersurface singularity (X,x) in the sense of Milnor* [4], *then*

$$(*)\qquad \begin{aligned}\mu&=\dim E_1^{n-1,1}+\dim E_2^{n,0}-\dim E_2^{n-1,0}\\&=\dim E_1^{1,n-2}+\dim E_2^{0,n-1}-\dim E_2^{1,n-1}.\end{aligned}$$