160. The Generalized Form of Poincaré's Inequality and its Application to Hypoellipticity

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Introduction. In this paper we shall derive an inequality of the form

(0.1) $\qquad \|u\| \leqq C(\zeta^{-\tau}\|u\|_\tau + \zeta^l \|gu\|) \qquad$ for $u \in C_0^\infty(B_{\delta_0})$, $\zeta > 0$

as an extended form of Poincaré's inequality, where B_{δ_0} is the open ball in R_x^n with the center $x=0$ and the radius $\delta_0>0$, τ is a positive number, and $g(x)$ is a real valued C^∞-function which vanishes of finite order l at the origin. If g is a homogeneous function satisfying $|g(x)| \geqq C_0|x|^l$ ($C_0>0$) we can easily derive (0.1) by deriving first an inequality $\|u\| \leqq C(\||D_x|^\tau u\| + \|gu\|)$ and using the homogeneity of g as in Grushin [2]. In the present paper using Hörmander's theorem in [4] we shall prove that the inequality (0.1) holds even in the case of non-homogeneous function $g(x)$.

As an application we shall prove the hypoellipticity for the operator of the form

(0.2) $\qquad L = a(X, D_x) + g(X)b(X, Y, D_y),$

when $a(x,\xi)$ satisfies the conditions similar to those in [3] and [7], $b(x,y,\eta)$ satisfies the conditions similar to those in the strongly elliptic case, and $g(x)$ is a non-negative function such that $\partial_x^{\alpha_0} g(0) \neq 0$ for some α_0. The idea of the proof is found in the proof of the hypoellipticity of the operator $Lu = |x|^2 \Delta_x^2(|x|^2 u) - \Delta_x u + i|x|^2 \Delta_y^3 u$ by Beals [1]. We note that the operator of the form (0.2) is a generalization of the operators $A(x; D_x) + g(x)^2 B(x, y; D_y)$ in Kato [5] and $(-\Delta_x)^l + |x|^{2\nu}(-\Delta_y)^{l'}$ in Grushin [2] and Taniguchi [8].

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§ 1. The generalized form of Poincaré's inequality. In this paper we shall use the following notations:

$$\partial_{x_j} = \partial/\partial x_j, \qquad j=1,\cdots,n,$$
$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n} \quad \text{for multi-index } \alpha=(\alpha_1,\cdots,\alpha_n),$$
$$\mathcal{B}(R_x^n) = \{u \in C^\infty(R_x^n);\ \sup_x |\partial_x^\alpha u(x)| < \infty \text{ for any } \alpha\},$$
$$\mathcal{S}(R_x^n) = \{u \in \mathcal{B}(R_x^n);\ x^\alpha \partial_x^\beta u \in \mathcal{B}(R_x^n) \text{ for any } \alpha, \beta\}.$$

Theorem 1. *Let $g(x) \in C^\infty(\overline{B_{\delta_0}})$ be a real valued function which satisfies for some α_0*