112. *Serial Endomorphism Rings*

By Ryohei MAKINO
Tokyo University of Education

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1. Recently, Ringel and Tachikawa [2] have proved that the endomorphism ring of a minimal generator cogenerator module over a serial ring is again serial. In this connection, the purpose of this note is to obtain a necessary and sufficient condition that the endomorphism rings of modules over a serial ring are serial.

Let R be a ring. An R-module M is said to be *serial* if its submodules form a finite chain. We call a ring R *left* (*right*) *serial* if ${}_RR$ (R_R) is a direct sum of serial modules. A left and right serial ring is called *serial*, and this is the same with a generalized uni-serial ring in the sense of Nakayama [1].

A subquotient U of an R-module M will be called *proper* if $U=A/B$ with $M\supsetneqq A\supseteq B\neq 0$, and we shall say that an R-module P appears as a proper subquotient of M if P is isomorphic to a proper subquotient of M. Subquotients U and V of a serial R-module M will be called *joined* if a non-zero submodule of one of U and V coincides with a non-zero factor module of the other of U and V. Let $M_1, \cdots, M_n$ be R-modules. An *iso*-subquotient of M_i will be a proper subquotient of M_i which is isomorphic to some M_j. A *pair*-subquotient of M_i will be a factor module of M_i which is isomorphic to a submodule of some M_j or a submodule of M_i which is isomorphic to a factor module of some M_j. With these definitions we can state the following main results.

Theorem 1. *Let R be a serial ring and $M_1, \cdots, M_n$ indecomposable left R-modules. The following statements are equivalent.*

a) *The endomorphism ring S of $M=M_1\oplus\cdots\oplus M_n$ is serial.*

b) *For each M_i, no iso-subquotient of M_i is joined with any pair-subquotient of M_i.*

As a special case, if there are no iso-subquotients, then the condition b) of Theorem 1 is satisfied, so we have

Corollary 1. *Let R be a serial ring and $M_1, \cdots, M_n$ indecomposable left R-modules. If no M_i appears as a proper subquotient of any M_j, then the endomorphism ring S of $M=M_1\oplus\cdots\oplus M_n$ is serial.*

Since each indecomposable module over a serial ring R is serial, no indecomposable injective or projective R-modules appear as a proper subquotient of any indecomposable R-modules. So, the above corollary can be regarded as a generalization of [2, Lemma 5.6].