4. On the Norm Properties on Function Spaces

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1. Introduction. Let X be a compact Hausdorff space and λ be a regular finite measure on X. Let $\phi_x(t)$ be a continuous strictly increasing function of $t \geqq 0$ for each $x \in X$ with $\phi_x(0)=0$ and $\lim_{t\to\infty} \phi_x(x) = +\infty$. We assume further that for a fixed $t>0$, the function $\phi_x(t)$ of $x \in X$ is always measurable and

(A) $$0 < \inf_{x\in X} \phi_x(t) \leqq \sup_{x\in X} \phi_x(t) < +\infty.$$

We define a so-called N-function: $\Phi_x(u)=\int_0^u \phi_x(t)dt,\ x \in X$. Then, we see easily that $\Phi_x(u)$ is a convex continuous function of $u \geqq 0$ for a fixed x and a measurable function of x for a fixed u. We shall consider the function space $L_{\Phi_x}(X)$ of measurable functions which is a so-called Orlicz-Nakano space. Since $\Phi_x(|f(x)|)$ is a non-negative measurable function of $x \in X$ for all measurable function f (with respect to λ) by assumption, we can define a functional

(B) $$M_{\Phi_x}(f)=\int_X \Phi_x(|f(x)|)d\lambda(x).$$

Let us define a function space of measurable functions

$L_{\Phi_x}(X)=\{f;$ measurable and $M_{\Phi_x}(cf)<+\infty$ for some $c>0\}$.

Now, we shall consider the complementary function $\Psi_x(u)$ for $\Phi_x(u)$ such that

$$\psi_x(t)= \sup_{\phi_x(u)\leqq t} u \qquad (=\phi_x^{-1}(t))$$

and

$$\Psi_x(u)=\int_0^u \psi_x(s)ds \qquad \text{for } x \in X.$$

We see by assumption $\psi_x(t)$ (resp. $\Psi_x(u)$) has the same properties as $\phi_x(t)$ (resp. $\Phi_x(u)$). In our discussion, $\|\cdot\|_{\Phi_x}$ means the norm defined in [3]. In [3], this norm is called the first modular norm.

Corresponding to an equi-measurable transformation in X, L_p and Orlicz spaces are of importance, since the norm of the function in these spaces is invariant under the transformation. But, in many cases which are not expected uniform properties at each point in X and will be occurred in applications, it is natural to consider the spaces $L_{\Phi_x}(X)$, since the property of functions may be changeable under the transformation. H. Nakano considered more wider sense than that of ours.