2. On Spaces with a Map $CP^n \to M$ of Degree One

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§ 1. **Introduction.** Let M be a connected oriented closed topological m manifold. It is known in [2] that if $f\ S^m \to M$ is a map of degree one, then f is a homotopy equivalence. And moreover L.E. Spence has proved in [3] that if a map $f\ S^p \times S^q \to M$ is of degree one then M has the homotopy type of S^{p+q} or f is a homotopy equivalence. In this note we shall consider the case of complex projective space. Then we shall prove

Theorem. *If M admits a map $f\ CP^n \to M$ of degree one, then M has the homotopy type of S^{2n}, CP^n, or cohomological quartenion projective space. Especially if n is odd M has the homotopy type of S^{2n} or CP^n.*

Corollary. *Let QP^n be the n dimensional quaternion projective space. If M admits a map $f\ QP^n \to M$ of degree one, then M has the homotopy type of S^{4n} or f is a homotopy equivalence.*

§ 2. **Some cohomological conditions.** At first we note the following lemma in [2]

Lemma 1. *Let M, N be connected oriented closed topological n manifold. If $f: M \to N$ is a degree one map, then*

(1) *$f_\# \pi_1(M) \to \pi_1(N)$ is an epimorphism.*

(2) *$f_* H_i(M) \to H_i(N)$ is a split epimorphism.*

(3) *$f^* H^i(M) \to H^i(N)$ is a monomorphism.*

Now let $f: CP^n \to M$ be a map of degree one. Then we obtain from Lemma 1 that M is simply connected and $H^i(M) \cong 0$ ($i=1$ mod. 2). Thus we may assume that $H^{2k}(M) \cong Z$, and $H^i(M) \cong Z$ $(0<i<2k)$.

Lemma 2. $n \equiv 0$ (mod. 2) *and* $H^*(M) = \dfrac{Z[\alpha]}{(\alpha^{n/k}+1)}$

Proof. Let α be a generator of $H^{2k}(M)=Z$, and μ_M be the fundamental class of $H^{2n}(M)$. By (3) of Lemma 1 we have $f\ (\alpha) = mx^k$ $(\mathrm{m} \neq 0)$ where x denotes the generator of $H^2(CP^n)$. Therefore, from $f\ (\alpha^s) = m^s x^{ks}$, we obtain that

$$H^{2i}(M) \cong Z,\ i=0 \text{ (mod. } k) \text{ and } i \leqq n.$$

Suppose that $n=ks+r$ $(0<r<k)$. Then by the duality of $H^*(M)$, we have $H^{2r}(M)=0$. But this contradicts the assumption. Thus we have $k=0$ (mod. n). Next we suppose that $H^{2a}(M)=Z$ $(jk<a<(j+1)k \leqq n$, for some $j)$ and let β be a generator of $H^{2a}(M)$. Then we have