90. *Paley-Wiener Type Theorem for the Heisenberg Groups*

By Shoichi ANDO
Department of Mathematics, Kyoto University

(Communicated by Kôsaku YOSIDA, M. J. A., Sept. 13, 1976)

1. The simply connected Heisenberg group G of n-th order consists of elements $g(x,y,z)$ $(x, y \in R^n, z \in R)$ with multiplication law $g(a,b,c)\cdot g(x,y,z)=g(x+a, y+b, z+c+\langle a, y\rangle)$, where $\langle a, y\rangle=\sum_{i=1}^n a_i y_i$.

In this paper we state a Paley-Wiener type theorem for the group G by the same method as in [3]. Let N and A be the subgroups of elements $n=g(0,b,c)$ and $a=g(a,0,0)$, respectively. Then $G=N\cdot A$ is a semidirect product. On the set $\hat{N}$ of not necessarily unitary characters of N co-adjoint action of A is defined by $a^*\cdot\chi(n)=\chi(ana^{-1})$, $(a \in A, \chi \in N)$. Every irreducible unitary representation of infinite dimension is realized up to equivalence in $L^2(R^n, dx)$ cf. [1], [2]: for $\lambda\neq 0$,

(1) $$T_g^\chi\varphi(g)=e^{\langle\mu,b\rangle}e^{\lambda(\langle b,x\rangle+c)}\varphi(x+a), \text{ for } g=g(a,b,c),$$

which is induced from a unitary character $\chi=(\mu,\lambda)$ of N such that $\chi(g(0,b,c))=\exp(\langle\mu,b\rangle+\lambda c)$, $(\mu\in\sqrt{-1}\cdot R^n, \lambda\in\sqrt{-1}\cdot R)$. Let $\mathcal{C}$ be the space of functions φ on R^n with finite seminorms $\|\cdot\|_t$ for any $t\in R^n$, where

$$\|\varphi\|_t=\left(\int_{R^n}\exp\langle t, |x|\rangle\cdot|\varphi(x)|^2\,dx\right)^{1/2}, \qquad (|x|=(|x_i|)_i).$$

In the space $(\mathcal{C}, \|\cdot\|_t)$ the formula (1) gives a representation $\mathcal{D}_\chi$. Especially we have $\|T_g^\chi\cdot\varphi\|_t\leqq C^\chi(t,g)\,\|\varphi\|_{\tau^\chi(t,g)}$, $(\varphi\in\mathcal{C})$, with constants $C^\chi(t,g)$ and $\tau^\chi(t,g)$ independent of φ. From easy argument of the existence of invariant bilinear forms follows

Proposition. (i) *A continuous linear operator commuting with all T_g^χ $(g\in G)$ is a scalar multiple of the identity.* (ii) *Representation $\mathcal{D}_\chi$ extends to a unitary one if and only if so is χ* (cf. [4]).

2. Let $Q_{\alpha,\beta,\gamma}$ be a compact set in G of the form

$$\{g(x,y,z)\,;\,|x_i|\leqq\alpha_i, |y_j|\leqq\beta_j, |z|\leqq\gamma, i, j=1\cdots n\}.$$

We assign auxiliary functions to $Q=Q_{\alpha,\beta,\gamma}$, $\tau^\chi(t\,;Q)=t+2\beta|Re\lambda|$, and $C^\chi(t,Q)=\exp[\langle\beta,|Re\mu|\rangle+\gamma|Re\lambda|+2^{-1}\langle|\tau^\chi(t\,;Q)|,\alpha\rangle]$.

Lemma. *If the support of a function $f\in L^1(G)$ is contained in the compact set Q, the Fourier transform of f: $T_f^\chi=\int_G f(g)T_g^\chi dg$, converges strongly in $\mathcal{C}$ for every $\chi\in\hat{N}$ and it holds*

(2) $$\|T_f^\chi\varphi\|_t\leqq C^\chi(t\,;Q)\,\|f\|_{L^1}\|\varphi\|_{\tau^\chi(t;Q)} \qquad (t\in R^n).$$

The Plancherel formula takes the following form: