17. Approximation of an Irrational Number by Rational Numbers.

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1. Let ω be a positive irrational number, $[a_0, a_1, a_2, \cdots\cdots]$ its expansion into the simple continued fraction, and

$$P_n/Q_n = [a_0, a_1, a_2, \cdots\cdots, a_n],$$
$$P_{\lambda, n}/Q_{\lambda, n} = [a_{\lambda+1}, \cdots\cdots, a_n].$$

Prof. FUJIWARA(1) proved that the minimum of

$$Q_n^2 \left| \omega - \frac{P_n}{Q_n} \right|, \quad Q_m^2 \left| \omega - \frac{P_m}{Q_m} \right| \quad \text{and} \quad Q_l^2 \left| \omega - \frac{P_l}{Q_l} \right|$$

is less than

$$\frac{1}{\sqrt{9 - \dfrac{4}{Q_{n,l}^2}}},$$

when $Q_{n,m}$, $Q_{m,l}$ and $Q_{n,l}$ satisfy MARKOFF's equation

$$x^2 + y^2 + z^2 = 3\,xyz$$

and $m-n$, $l-m$ are odd.

By his suggestion, I have determined the numbers m, l for any MARKOFF's period.

2. Adopting MARKOFF's notations(2), let $Q\{a, a_1, a_2, \cdots\cdots, a_k, 2\}$ and $\mathfrak{R}\ (2, a, a, \cdots\cdots \lambda, \lambda, 2)$ be MARKOFF's number and the period of the continued fraction corresponding to the period $\{a, a_1, a_2, \cdots\cdots a_k, 2\}$ respectively. Then from the relation

$$\{a+1, a_1, \cdots\cdots, a_k, 2\} = \{a, a_1, a_2, \cdots\cdots, a_k, 2\} + \{a_1 - 1, a_2, \cdots\cdots a_k, 2\}\ (k = \text{odd}),$$
$$\text{or} \qquad = \{a_1 - 1, a_2, \cdots, a_k, 2\} + \{a, a_1, a_2, \cdots\cdots a_k, 2\}\ (k = \text{even}),$$

(1) These Proceedings 2, 1–3.

(2) A. MARKOFF, Sur les formes quadratiques binaires, Math. Ann., 17 (1880), or Bachmann, Die Arithmetik der quadratischen Formen II, 106–129.