34. *A New Elementary Proof of a Theorem of Minkowski.*

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In the following Note, No. 35, Fukasawa extends Klein's geometrical interpretation of continued fraction and proves Minkowski's theorem[1] in a more precise form. I will here add another simple proof based on the same standpoint as in my previous paper on the approximation of an irrational number by rational numbers.[2]

Let α be any positive irrational number, β any real number between 0 and 1, for which there is no pair of integers (x, y) which satisfies $\alpha x-y+\beta=0$. Further let L be the straight line $\alpha x-y+\beta=0$ and $A_0=(0,0)$, $B_0=(-1,0)$. Then construct two polygonal lines $(A)=A_0A_1A_2\cdots$, $(B)=B_0B_1B_2\cdots$, convex towards L, such that their vertices are all lattice points, that is points whose coordinates are integers, and that there is no lattice point in the domain D enclosed by the x-axis, (A) and (B).

Let C_{n+1} be any lattice point on (A) or (B), say (B), and C_n, C_{n+2} two consecutive lattice points on (A), such that the abscissa of C_{n+1} lies between those of C_n and C_{n+2}. If we construct the parallelogram $C_{n-1}C_n$ C_{n+1} C_{n+1}, then C_{n-1} must lie below the x-axis; for, the straight line passing through C_{n+1}, parallel to C_nC_{n+2} cuts the x-axis at a point M between A_0, B_0, and if $C_{n+1}M<C_nC_{n+1}$, then there will be a lattice point on the segment $C_{n+1}M$, which lies in the domain D, contrary to the assumption.

Let the coordinates of C_k be (Q_k, P_k), and M_k be the intersection of L with the line passing through C_k parallel to the y-axis, then $\alpha Q_k-P_k+\beta$ is equal to C_kM_k with the sign + or − according as C_k lies above or below L. Therefore from the assumption we have

1) Minkowski, Diophantische Approximationen. See also Remak, Neuer Beweis eines Minkowskischen Satzes, Journal f. Math., 142 (1913); Scherrer, Zur Geometrie der Zahlen, Math. Annalen, 89 (1923).

2) These Proceedings, 2, 1 – 3.