83. *On the Zero Points of a Bounded Analytic Function.*

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1. Let

$$f(x) = 1 + a_1x + \cdots + a_nx^n + \cdots\cdots \tag{1}$$

be regular for $|x| < 1$ and x_0 the zero point of $f(x)$ which has the smallest modulus. Without any further restriction on $f(x)$, there does not exist a positive constant ρ, such that $|x_0| \geqq \rho$ for all functions (1), as the following example shows: in fact $f(x) = 1 + nx$ has a root $x = -\dfrac{1}{n}$, whose modulus can be made very small by taking n sufficiently large. Now we impose on $f(x)$ the restriction:

$$f(0) = 1, \quad \text{and} \quad |f(x)| < M \text{ for } |x| < 1^{1)} \tag{2}$$

For brevity, we call such a function $f(x)$ *a function of class M*, and will prove the existence of a positive quantity ρ_n, which has the following properties:

1) Every function of class M has at most $n-1$ roots in the circle $|x| < \rho_n$.

2) Among the functions of class M, there exists a function which has just n roots in the circle $|x| \leqq \rho_n$.

As we shall see later, the number of roots of a function of class M in the circle $|x| \leqq \rho_n$ can not exceed n and when there are just n roots in the circle $|x| \leqq \rho_n$ all roots must lie on the circle $|x| = \rho_n$. We will call such a function of class M that has just n roots on the circle $|x| = \rho_n$ an *extremal* function.

Theorem I. Let ρ_n be the quantity defined above, then

$$\rho_n = \frac{1}{\sqrt[n]{M}},$$

the extremal functions are $f(x) = \dfrac{a_1 - x}{1 - \bar{a}_1x} \cdot \dfrac{a_2 - x}{1 - \bar{a}_2x} \cdot \cdots \dfrac{a_n - x}{1 - \bar{a}_nx} \cdot \dfrac{1}{a_1 \cdots a_n}$,

1) From $f(0) = 1$ and $|f(x)| < M$ we have $M > 1$.