101. *On the System of Generalized Orthogonal Functions and its Relation to the Singular Integral Equations.*

By Matsusaburô FUJIWARA, M.I.A.

Mathematical Institute, Tohoku Imperial University, Sendai.

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Let $\varphi(x, \lambda)$ be a real or complex continuous function of a real variable x in the interval $(0,\infty)$, and of a real parameter λ in a certain interval, such that $M\{\varphi(x, \lambda)\,\overline{\varphi}(x, \mu)\} = 0$ or 1, according as $\lambda \neq \mu$ or $\lambda = \mu$, where $\overline{\varphi}$ denotes the conjugate complex function of φ, and $M\{f(x)\}$ means after H. Bohr

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(x)\,dx.$$

We call the family of such functions the system of generalized orthogonal functions. We restrict ourselves in the following to the real function $\varphi(x, \lambda)$.

If $f(x)$ be a real continuous function for $x \geqq 0$, for which $M\{f^2(x)\}$ exists and >0, and further if it can be uniformly approximated by finite number of functions belonging to the system $\{\varphi(x, \lambda)\}$, i.e. for any $\varepsilon>0$ there exist an integer N and two sequences of real numbers $(a_1, a_2, \cdots\cdots, a_N)$, $(\mu_1, \mu_2, \cdots\cdots, \mu_N)$, such that

$$\left| f(x) - \sum_1^N a_k \varphi(x, \mu_k) \right| < \varepsilon \text{ for all } x \geqq 0,$$

then Bohr's theory of the almost periodic functions may be extended to this case. For example, we can prove that for such a function $f(x)$ there corresponds an at most enumerable set of real numbers $(\lambda_1, \lambda_2, \cdots)$, for which $M\{f(x)\,\varphi(x, \lambda_k)\} = A_k \neq 0$, while for any other values of λ, $M\{f(x)\,\varphi(x, \lambda)\} = 0$, and further that $\sum A_k\,\varphi(x, \lambda_k)$ converges in means to $f(x)$, i.e.

$$\lim_{N\to\infty} M\left\{ f(x) - \sum_1^N A_k \varphi(x, \lambda_k) \right\} = 0,$$

or

$$M\left\{ f^2(x) \right\} = \sum_1^\infty A_k^2.$$